2D-DSP

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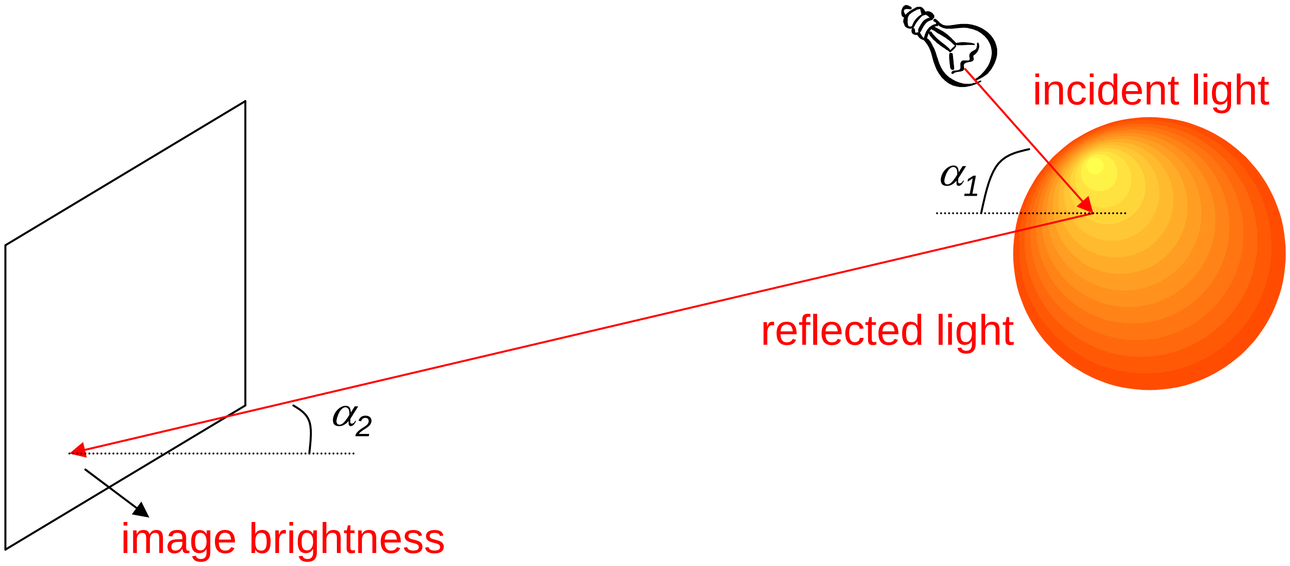
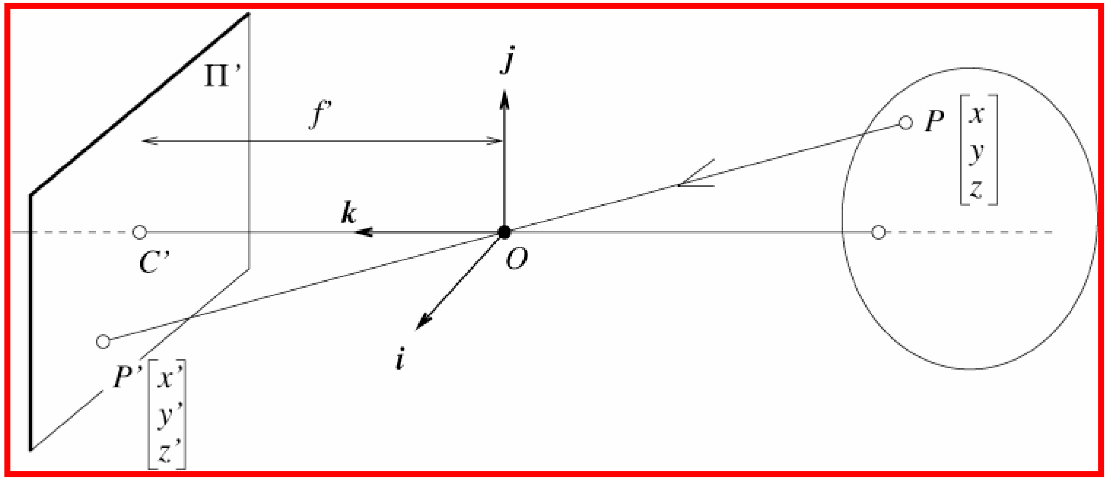
Image formation

- we have been studying the process of image formation
- three questions
  - what 3D point projects into pixel (x,y)?
  - what is the light incident on the pixel?
  - what is the pixel color?
- these determine the image value at the pixel



Geometry

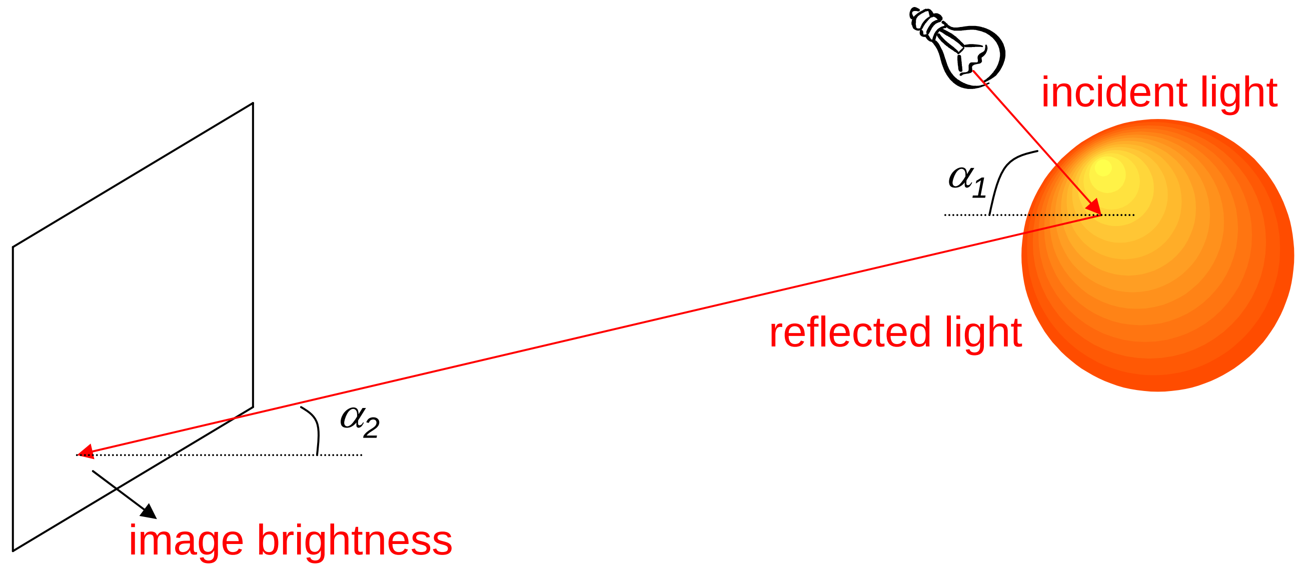
- geometry answers the first question
- pinhole camera:
  - point (x,y,z) in 3D scene projected into image pixel of coordinates (x', y')
  - according to the perspective projection equation:



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f \begin{pmatrix} x/z \\ y/z \end{pmatrix}$$

Light

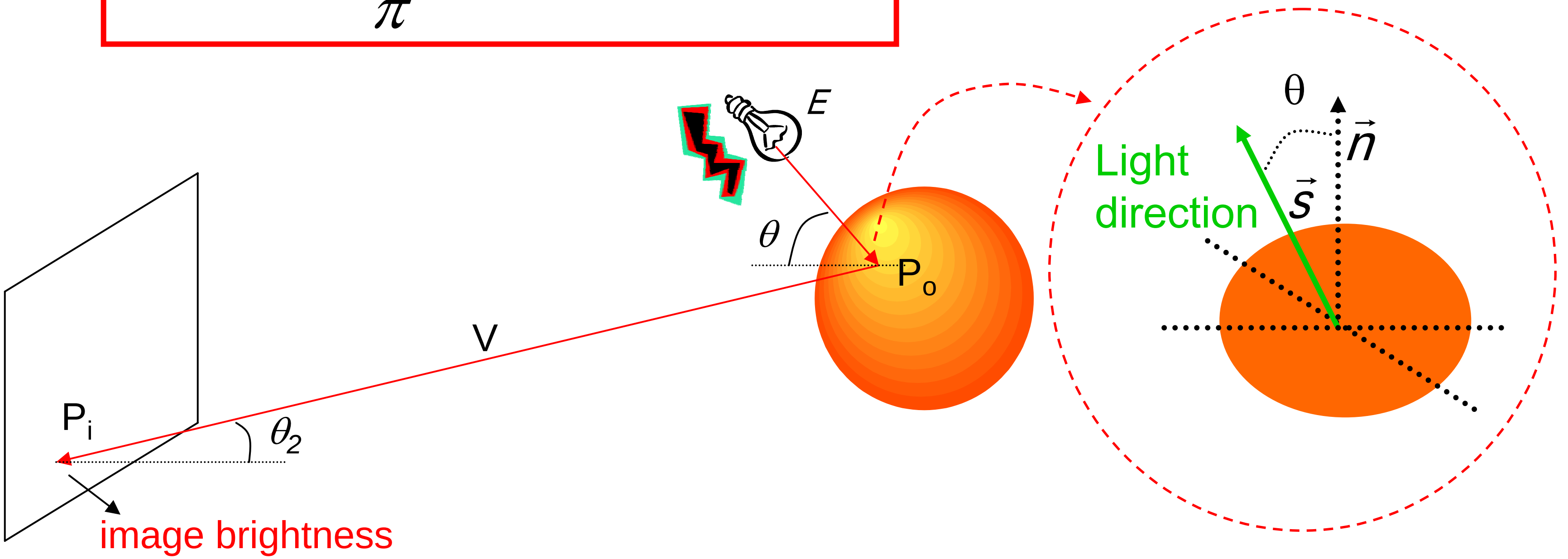
- the second depends on three main factors:
  - lighting of the scene
  - the reflectance properties of the material
  - various angles with which the light bounces from the objects



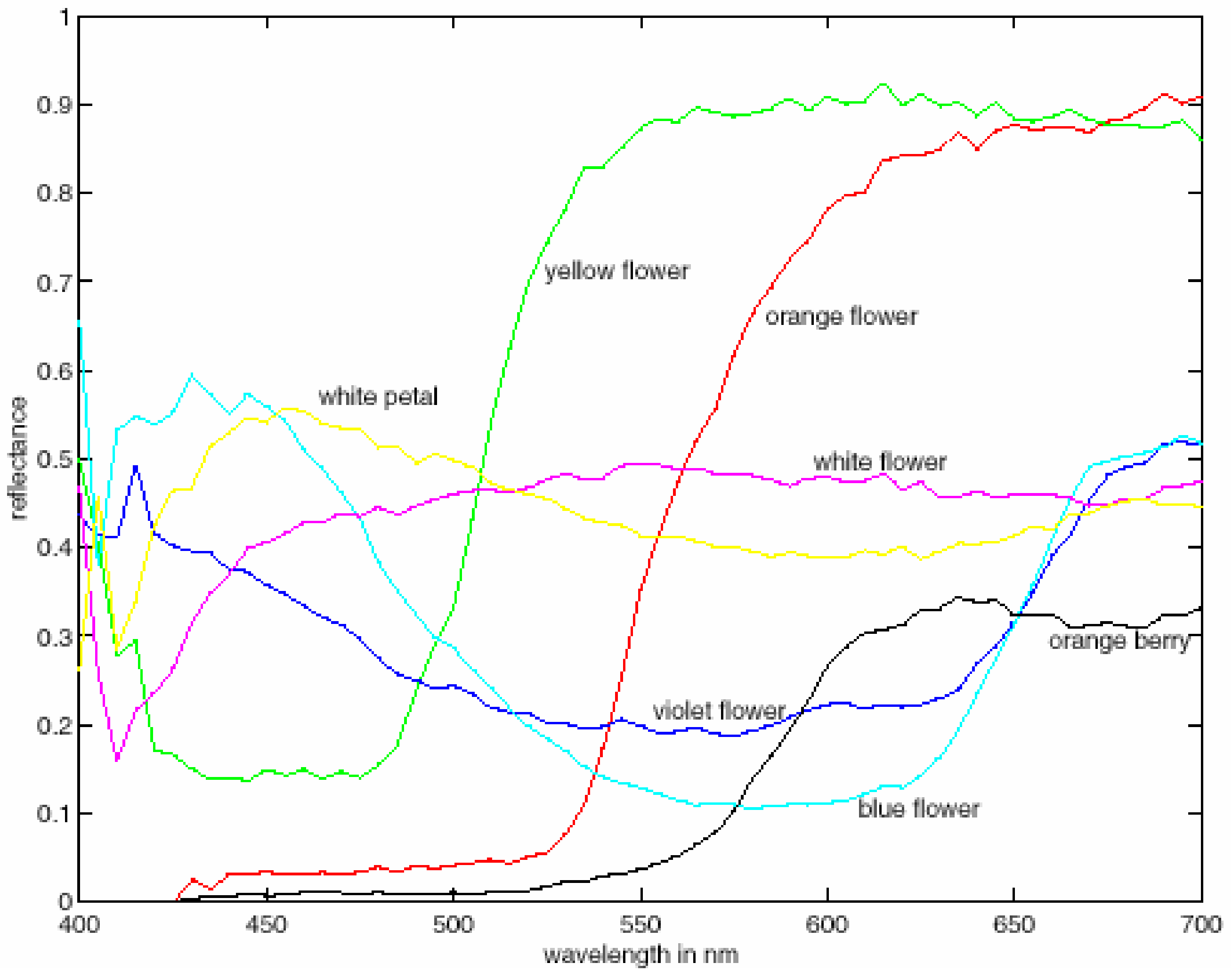
Lambertian surfaces

- we have a very simple equation
  - when surface is Lambertian and source a PS @ infinity
  - "image power = source power x object albedo x cos(light direction, surface normal)"

$$P(P_i) = \frac{E}{\pi}\rho_a(P_0)\vec{n}(P_0).\vec{s}$$



Spectral albedos

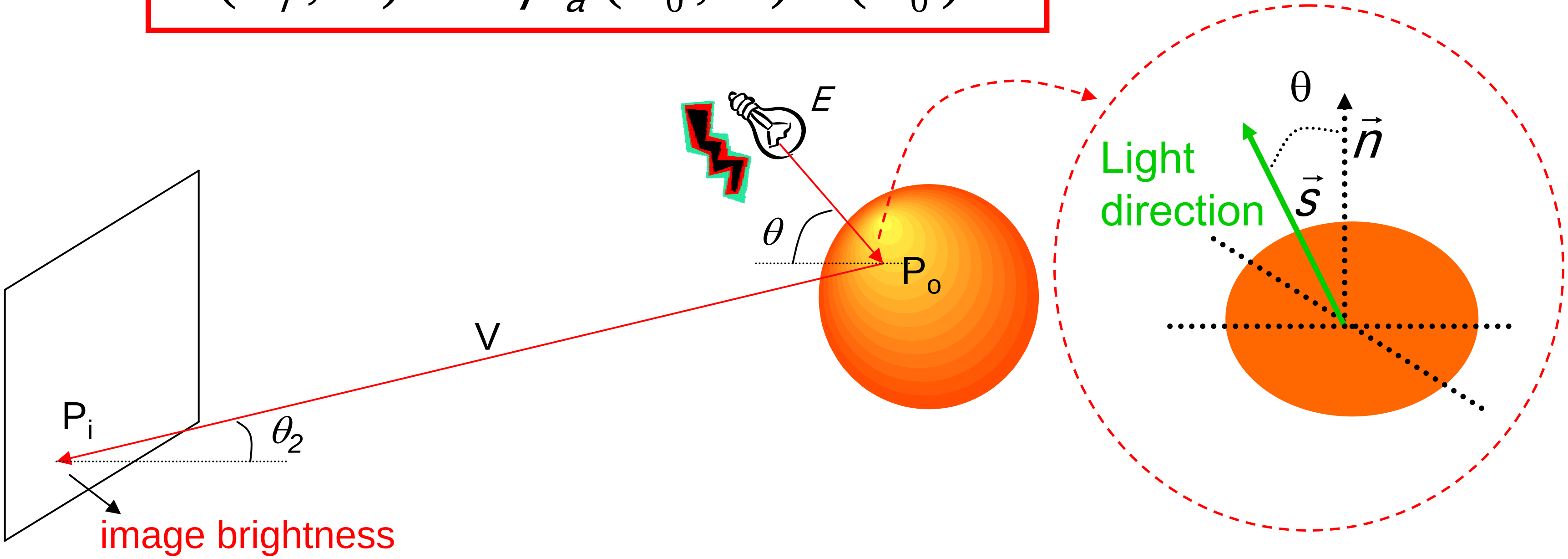


- with color, everything is replicated at each wavelength
- different objects have different spectral albedo, and that is why we perceive color

Spectral albedo

- how does this change our radiometry equation?
  - ***n*** (surface normal), ***s*** (light direction), do not change with wavelength
  - the dependence on wavelength can come from ρ the surface albedo, or *E* the source power

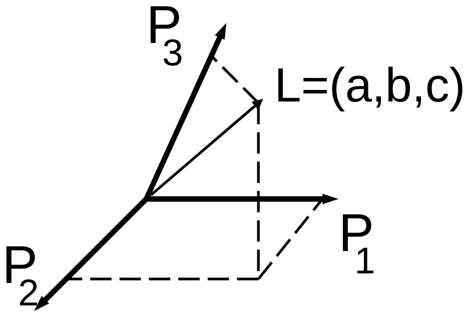
$$P(P_i, \lambda) = E\rho_a(P_0, \lambda)\vec{n}(P_0).\vec{s}$$



Color spaces

- color can be represented in different color spaces
  - a color space is defined by a set of three primaries

$$\{P_0(\lambda), P_1(\lambda), P_2(\lambda)\}$$



  - any color is a linear combination of these

$$L(\lambda) = aP_0(\lambda) + bP_1(\lambda) + cP_2(\lambda)$$

  - the coordinates are found by projection onto the matching functions

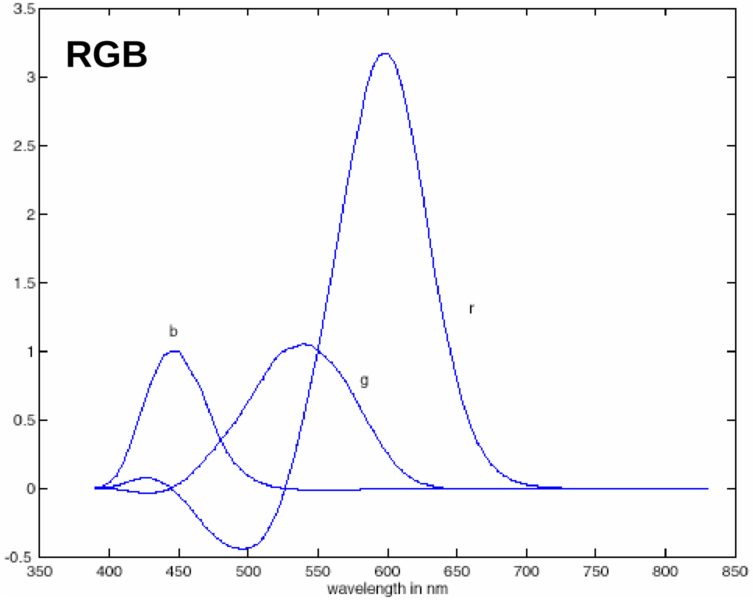
$$a = \int L(\lambda_0) f_1(\lambda_0) d\lambda_0; \quad b = \int L(\lambda_0) f_2(\lambda_0) d\lambda_0$$

$$c = \int L(\lambda_0) f_3(\lambda_0) d\lambda_0$$

  - the matching functions are the solutions of

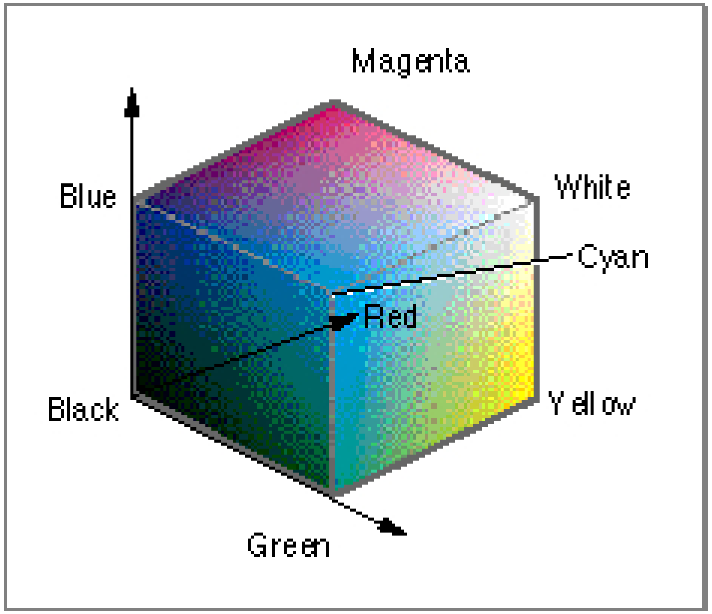
$$\delta(\lambda - \lambda_0) = \sum_{i=1}^{3} f_i(\lambda_0) P_i(\lambda)$$

Matching functions



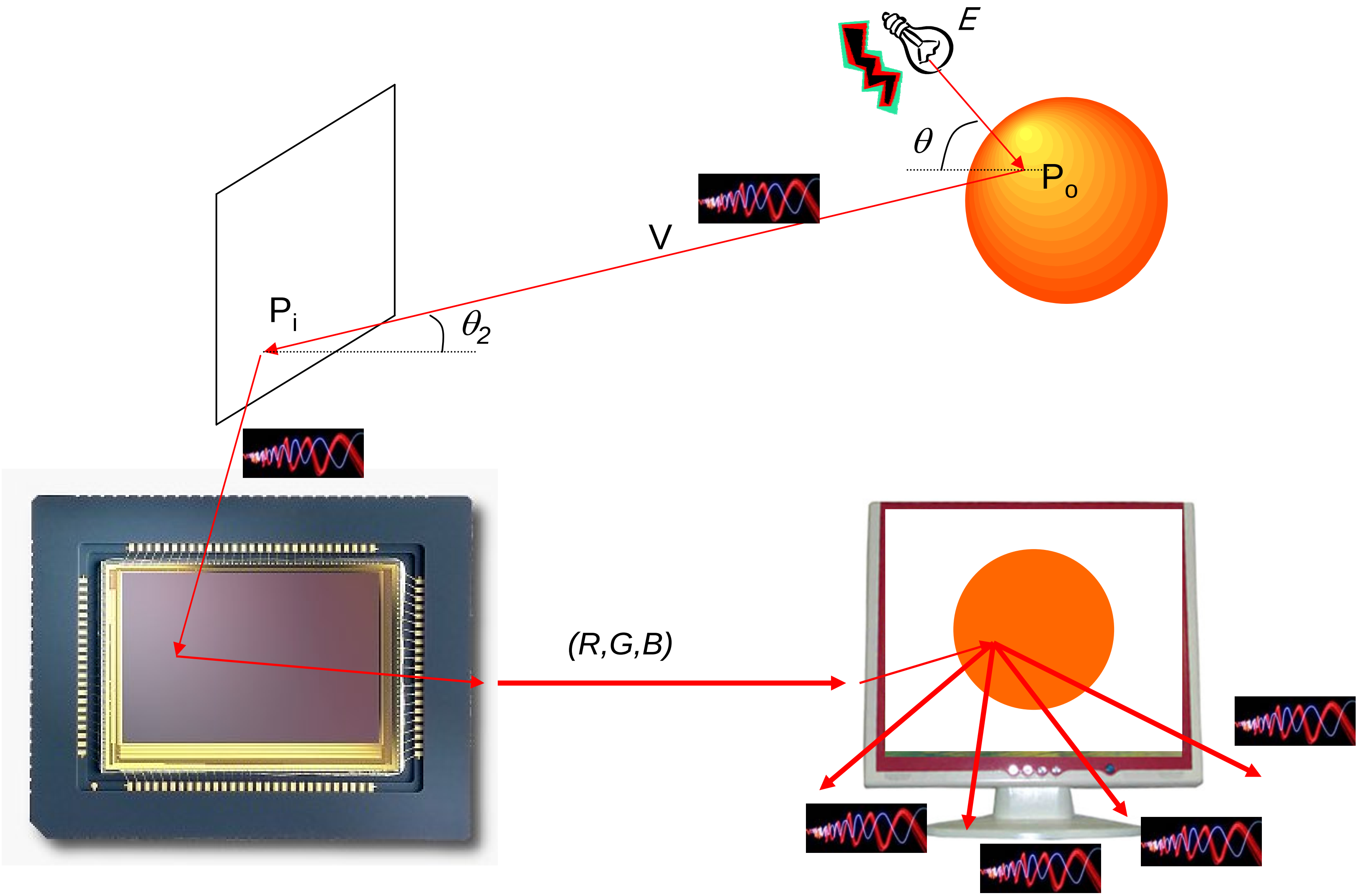
$$R = \int L(\lambda_0) r(\lambda_0) d\lambda_0;$$
$$G = \int L(\lambda_0) g(\lambda_0) d\lambda_0$$
$$B = \int L(\lambda_0) b(\lambda_0) d\lambda_0$$

- after the projection:
  - color is represented by three numbers
  - since primaries are known these are all that needs to be stored



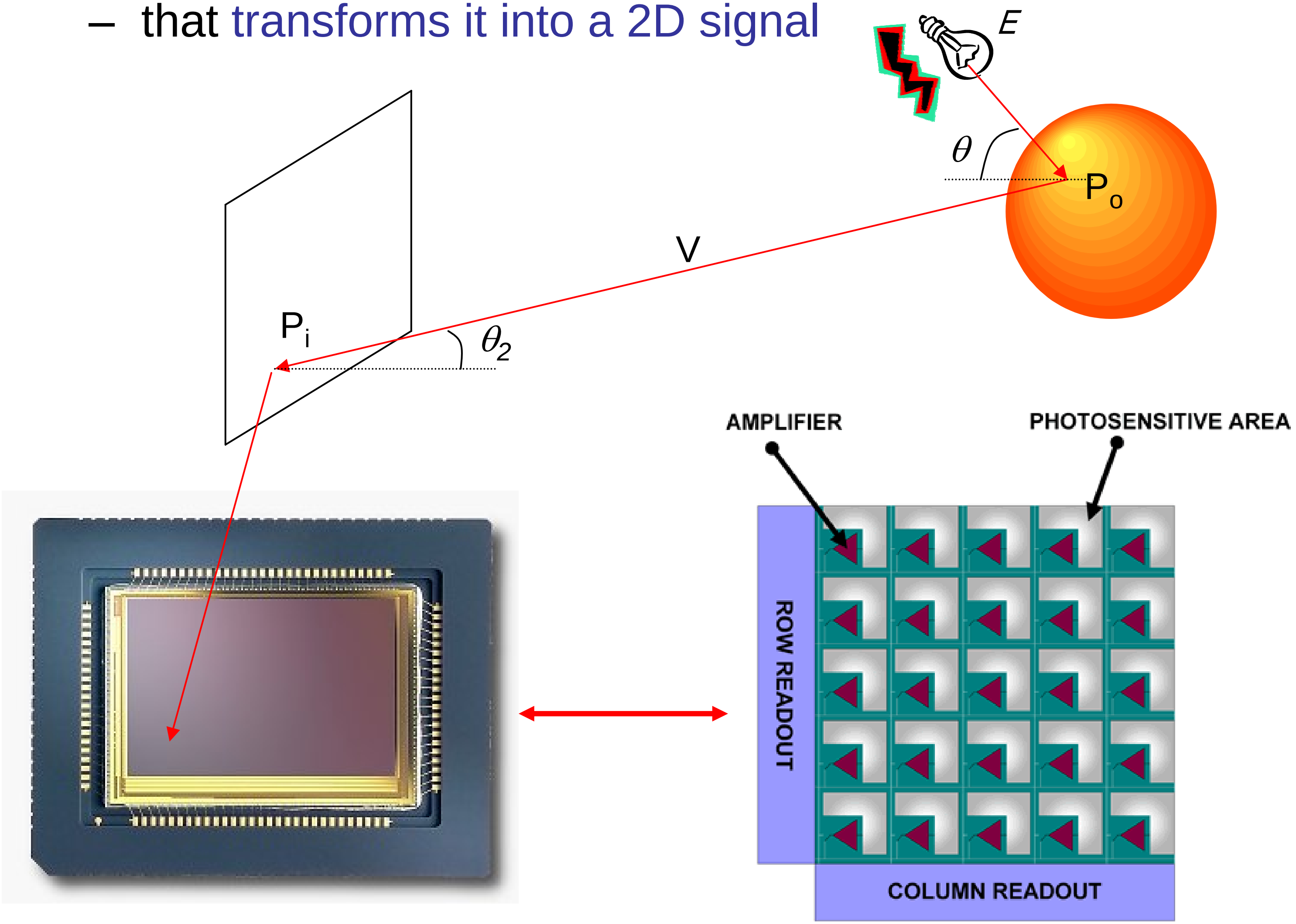
Images

- this summarizes the process



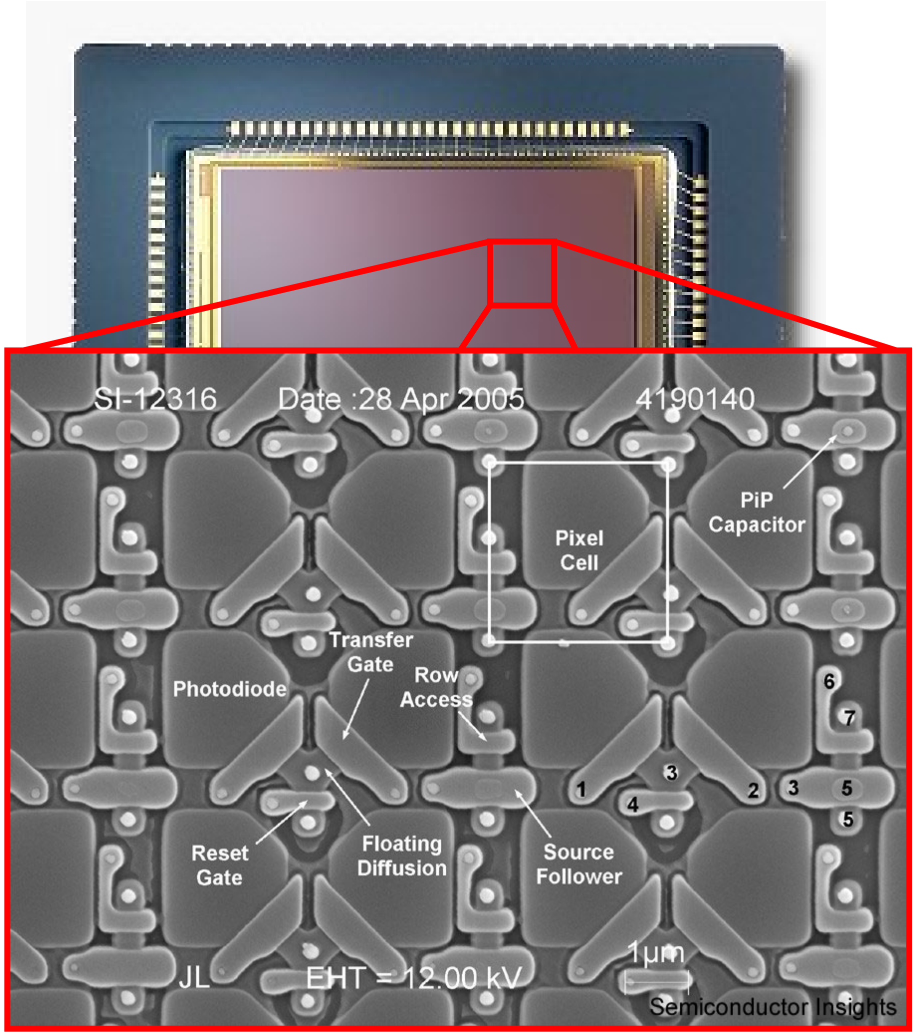
Images

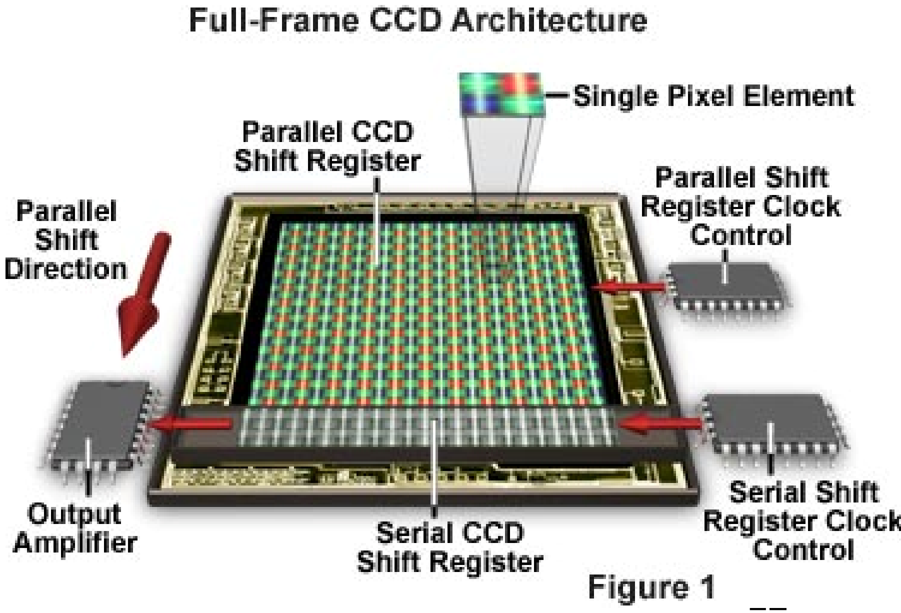
- the incident light is collected by an image sensor
  - that transforms it into a 2D signal



Imaging

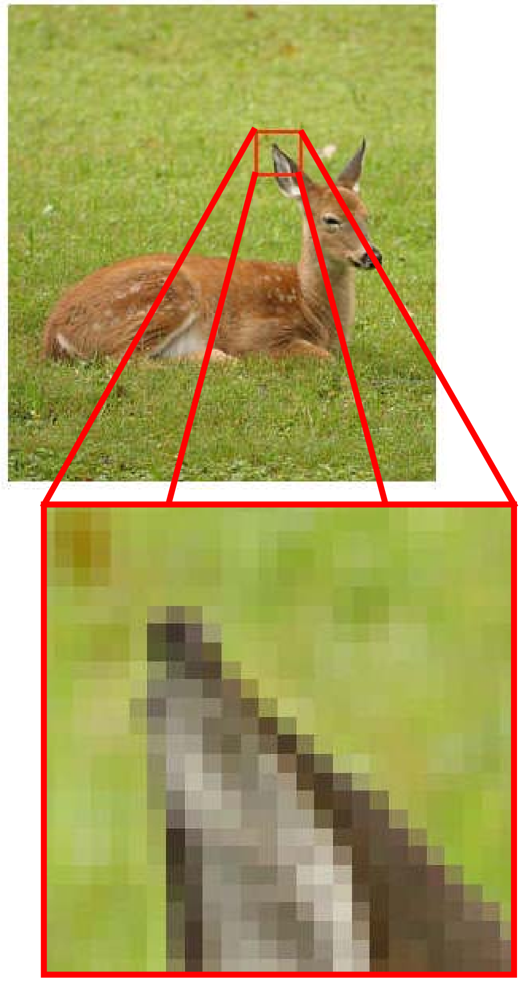
- the sensor is a 2D array of photosensitive cells
  - each cell captures the three color components of a picture element, or pixel
  - output: 2D pixel array



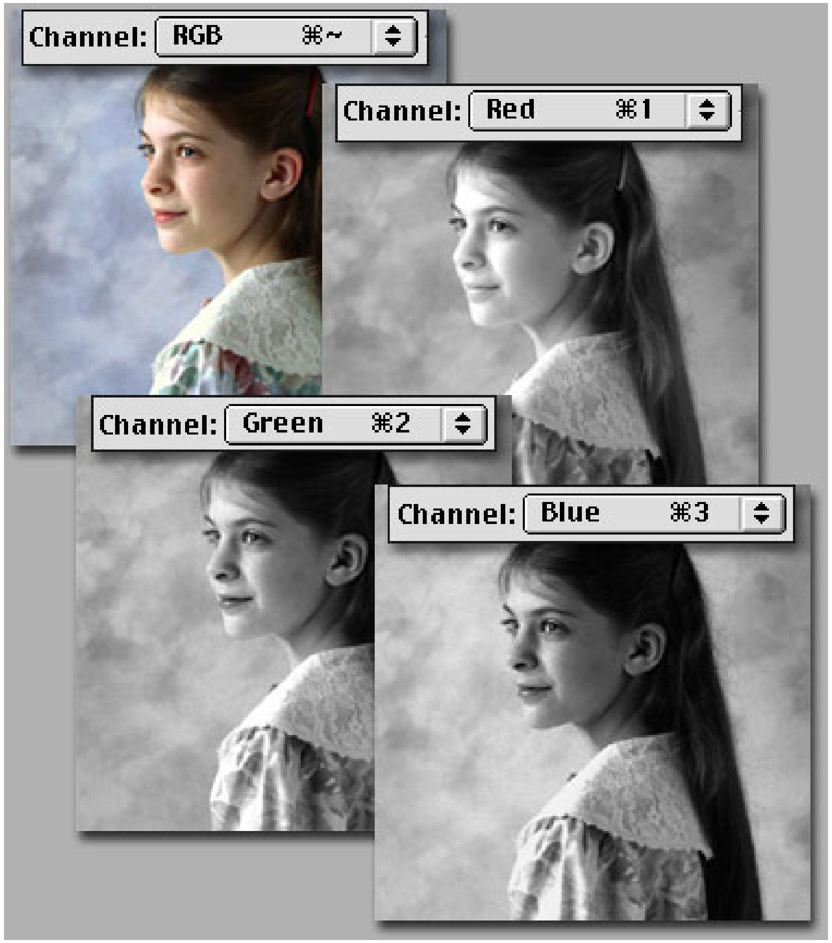


Digital images

- the image we see is this array



- composed of three color channels



2D-DSP

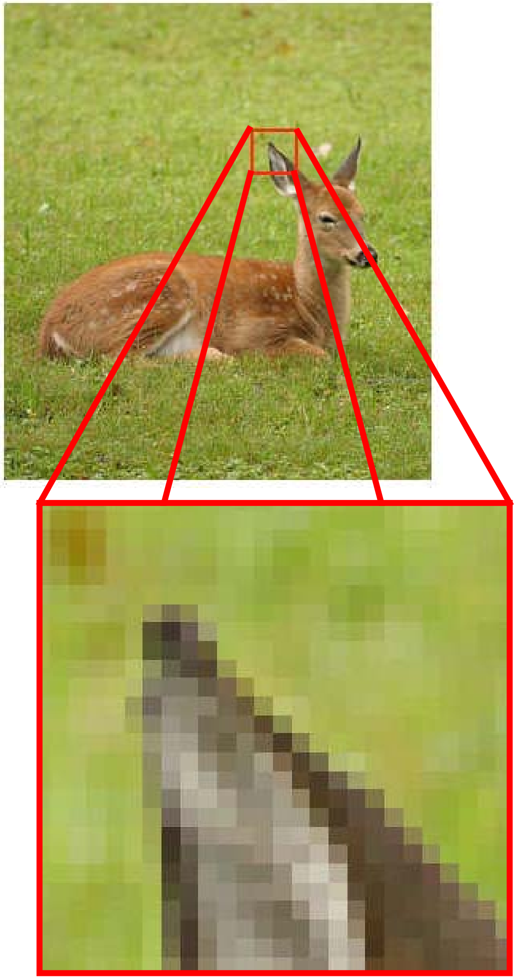
- in summary:
  - image is a *N x M* array of pixels
  - each pixel contains three colors
  - overall, the image is a 2D discrete-space signal
  - each entry is a 3D vector

$$x[n_1, n_2] = (r, g, b), \quad n_1 \in \{0, \ldots, N\}$$
$$n_2 \in \{0, \ldots, M\}$$

  - for simplicity, we consider only single channel images

$$x[n_1, n_2], \quad n_1 \in \{0, \ldots, N\}$$
$$n_2 \in \{0, \ldots, M\}$$

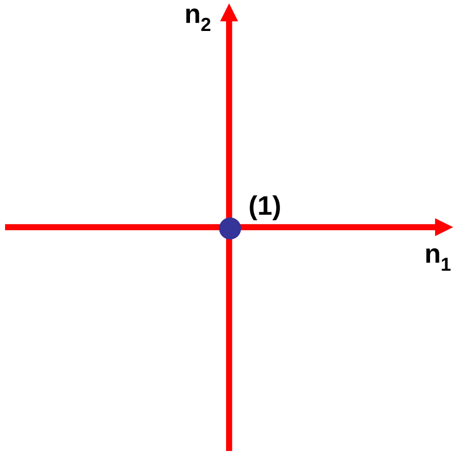
  - but everything extends to color in a straightforward manner



Important sequences

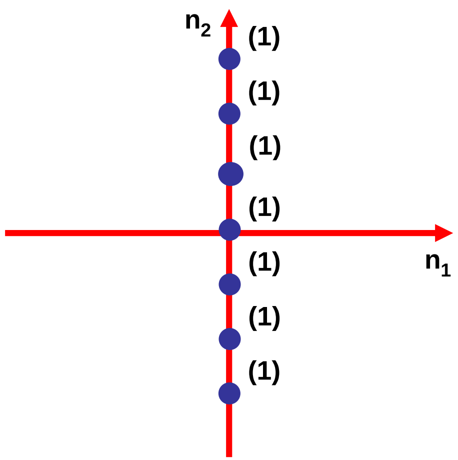
- impulse

$$\delta[n_1,n_2]=\begin{cases}1 & n_1=n_2=0\\ 0 & otherwise\end{cases}$$



- line impulses

$$x[n_1,n_2]=\delta_T(n_1)=\begin{cases}1 & n_1=0\\ 0 & otherwise\end{cases}$$



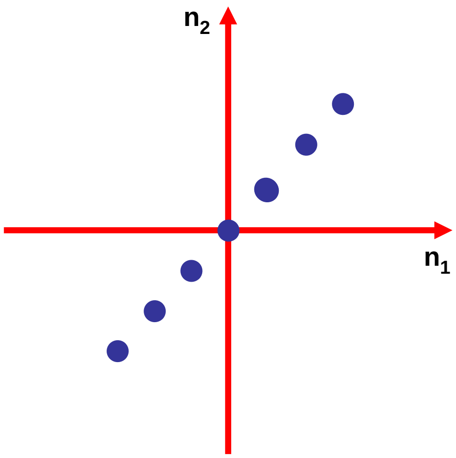
  - note that this is a 1D signal, embedded in the 2D plane
  - this is what the T subscript indicates
  - makes clear that we are not talking about the 1D delta

$$\delta[n_1]=\begin{cases}1 & n_1=0\\ 0 & otherwise\end{cases}$$

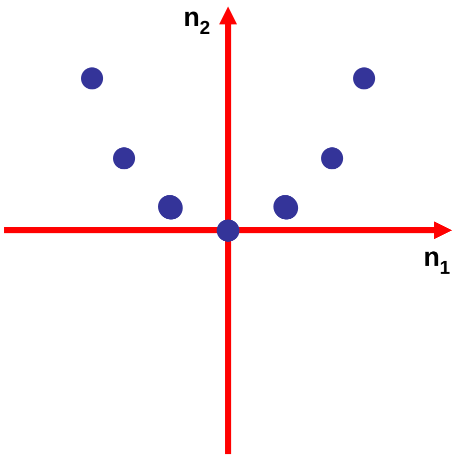
Important sequences

- unlike 1D, there are many 1D impulses
  - e.g.

$$\delta_T[n_1 - n_2]$$



$$\delta_T[n_2 - n_1^2]$$



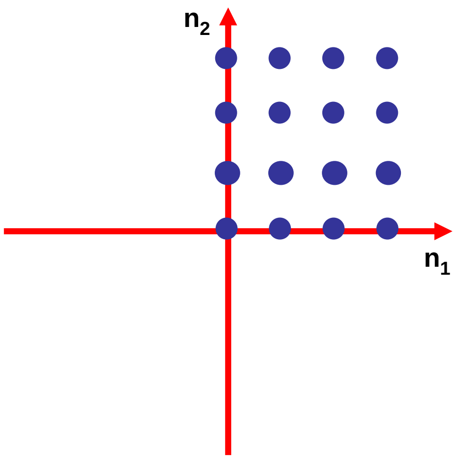
  - note: when the amplitude is 1 we omit the (1)

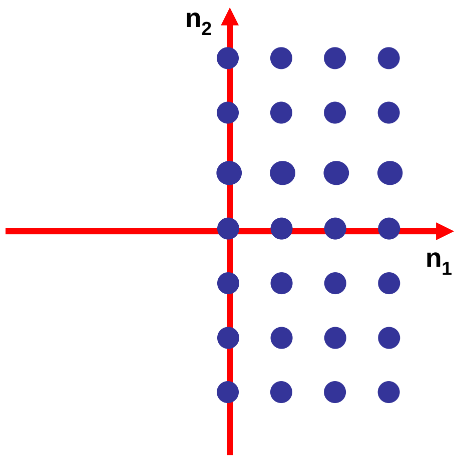
Important sequences

- step sequences

$$u[n_1, n_2] = \begin{cases} 1 & n_1, n_2 \geq 0 \\ 0 & otherwise \end{cases}$$

$$u_T[n_1] = \begin{cases} 1 & n_1 \geq 0 \\ 0 & otherwise \end{cases}$$





- exponential sequences: sequences of the type

$$x[n_1, n_2] = A\alpha^{n_1}\beta^{n_2}$$

Separable sequences

- a trivial concept,
  - but probably the only real novelty in this lecture
  - very important in practice, because it reduces 2D problem to collection on 1D problems

- **Definition:** a sequence is separable if and only if

$$x[n_1, n_2] = f[n_1] \times g[n_2]$$

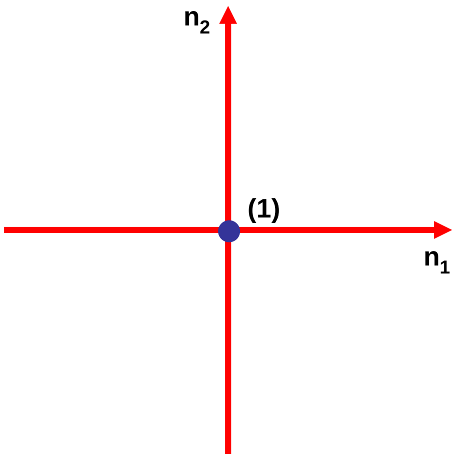
where *f[.]* and *g[.]* are 1D functions

- note: there are many examples of separable sequences
- but most sequences are not separable

Separable sequences

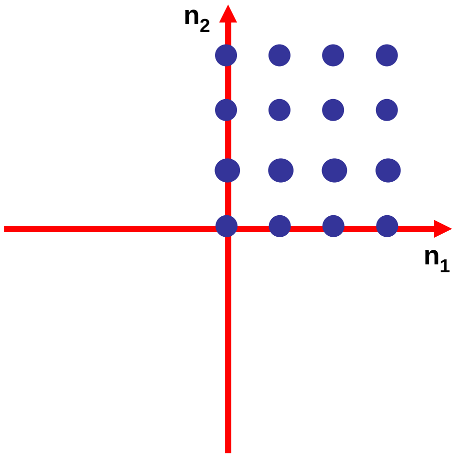
- impulse

$$\delta[n_1,n_2] = \begin{cases} 1 & n_1 = n_2 = 0 \\ 0 & otherwise \end{cases}$$
$$= \begin{cases} 1 & n_1 = 0 \\ 0 & o.w. \end{cases} \times \begin{cases} 1 & n_2 = 0 \\ 0 & o.w. \end{cases}$$
$$= \delta[n_1] \times \delta[n_2]$$



- step

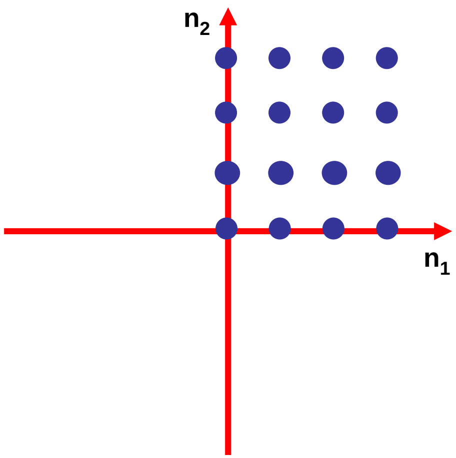
$$u[n_1,n_2] = \begin{cases} 1 & n_1, n_2 \geq 0 \\ 0 & otherwise \end{cases}$$
$$= \begin{cases} 1 & n_1 \geq 0 \\ 0 & o.w. \end{cases} \times \begin{cases} 1 & n_2 \geq 0 \\ 0 & o.w. \end{cases}$$
$$= u[n_1] \times u[n_2]$$



Separability

- in general, how do I know that sequence is separable?
- think of $x[n_1,n_2]$ as a matrix

$$X = \begin{bmatrix} & \vdots & \\ \cdots & x_{i,j} & \cdots \\ & \vdots & \end{bmatrix}, \text{ with } x_{i,j} = x[j,i]$$



- the condition $x[n_1,n_2] = f[n_1] \times g[n_2]$ is equivalent to

$$X = \begin{bmatrix} & \vdots & \\ \cdots & x_{n_2,n_1} & \cdots \\ & \vdots & \end{bmatrix} = \begin{bmatrix} \vdots \\ g_{n_2} \\ \vdots \end{bmatrix} . \begin{bmatrix} \cdots & f_{n_1} & \cdots \end{bmatrix} = gf^T$$

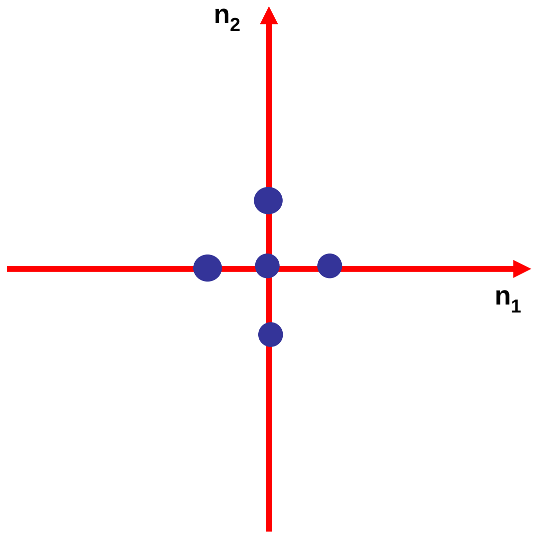
- this matrix has special properties!

Separability

- for a matrix $X = g \cdot f^T$
  - all rows are multiples of each other
  - the matrix has rank 1
  - only one eigenvalue is different than zero
- by testing any of these properties, you can check seprability
  - e.g. is this separable?
  - the matrix form is

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

  - the rank is 2, not separable!



Linear Shift Invariant (LSI) systems

- straightforward extension of LTI systems

- **Definition:** a system T that maps $x[n_1,n_2]$ into $y[n_1,n_2]$ is LSI if and only if
  - it is linear

$$
\begin{aligned}
T\{ax_1[n_1,n_2]+bx_2[n_1,n_2]\} &= \\
&= aT\{x_1[n_1,n_2]\}+bT\{x_2[n_1,n_2]\} \\
&= ay_1[n_1,n_2]+by_2[n_1,n_2]
\end{aligned}
$$

  - it is shift invariant

$$T\{x[n_1-m_1,n_2-m_2]\}=y[n_1-m_1,n_2-m_2]$$

Linear Shift Invariant (LSI) systems

- example: is $T\{x[n_1,n_2]\} = x^2[n_1,n_2]$ LSI?
  - invariance?

$$
\begin{aligned}
T\{x[n_1-m_1,n_2-m_2]\} &= x^2[n_1-m_1,n_2-m_2] \\
&= y[n_1-m_1,n_2-m_2]
\end{aligned}
$$

  - linearity?

$$
\begin{aligned}
T\{ax_1[n_1,n_2]+bx_2[n_1,n_2]\} &= \\
&= a^2x_1^2[n_1,n_2]+b^2x_2^2[n_1,n_2]+2abx_1[n_1,n_2]x_2[n_1,n_2] \\
&\neq ax_1^2[n_1,n_2]+bx_2^2[n_1,n_2] \\
&= ay_1[n_1,n_2]+by_2[n_1,n_2]
\end{aligned}
$$

  - the system is shift invariant but not linear

Linear Shift Invariant (LSI) systems

- example: is $T\{x[n_1,n_2]\} = g[n_1,n_2] \cdot x[n_1,n_2]$ LSI?
  - linearity?

$$
\begin{aligned}
T\{ax_1[n_1,n_2] + bx_2[n_1,n_2]\} &= \\
&= ag[n_1,n_2]x_1[n_1,n_2] + bg[n_1,n_2]x_2[n_1,n_2] \\
&= ay_1[n_1,n_2] + by_2[n_1,n_2]
\end{aligned}
$$

  - invariance?

$$
\begin{aligned}
T\{x[n_1-m_1,n_2-m_2]\} &= \\
&= g[n_1,n_2]x[n_1-m_1,n_2-m_2] \\
&\neq y[n_1-m_1,n_2-m_2]
\end{aligned}
$$

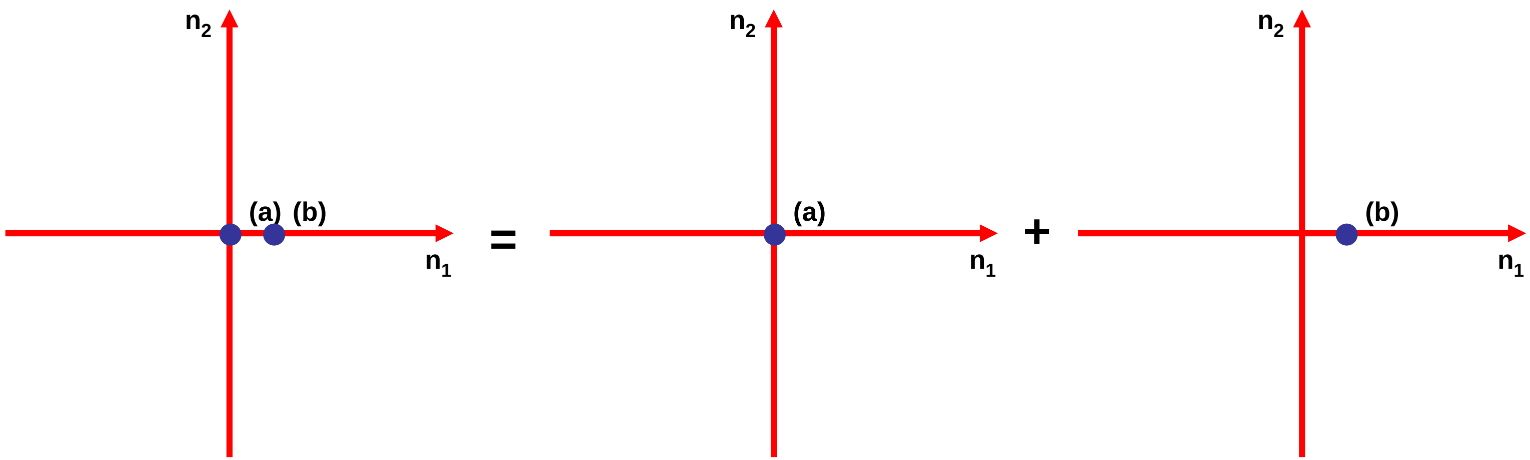
  - the system is linear but not shift invariant

Linear Shift Invariant (LSI) systems

- why do we care about LSI systems?
  - any signal can be written as

$$x[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\delta[n_1 - k_1, n_2 - k_2]$$

  - e.g.



$$a\delta[n_1, n_2] + b\delta[n_1 - 1, n_2]$$

$x[0,0]$ $x[1,0]$

Linear Shift Invariant (LSI) systems

- why do we care?
  - for an LSI system, this means that

$$y[n_1, n_2] = T\left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\delta[n_1 - k_1, n_2 - k_2] \right\}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, T\{\delta[n_1 - k_1, n_2 - k_2]\}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]$$

  - where

$$h[n_1, n_2] = T\{\delta[n_1, n_2]\}$$

is the impulse response of the system

2D convolution

- the operation

$$y[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]$$

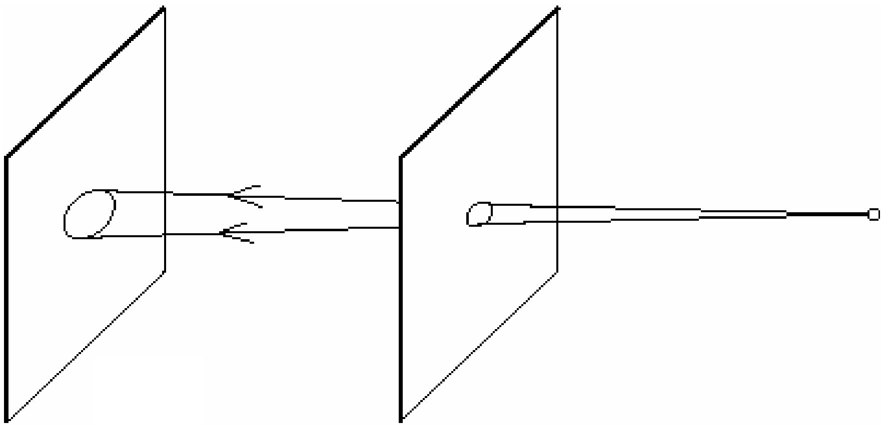
  is the 2D convolution of x and h
  - we will denote it by

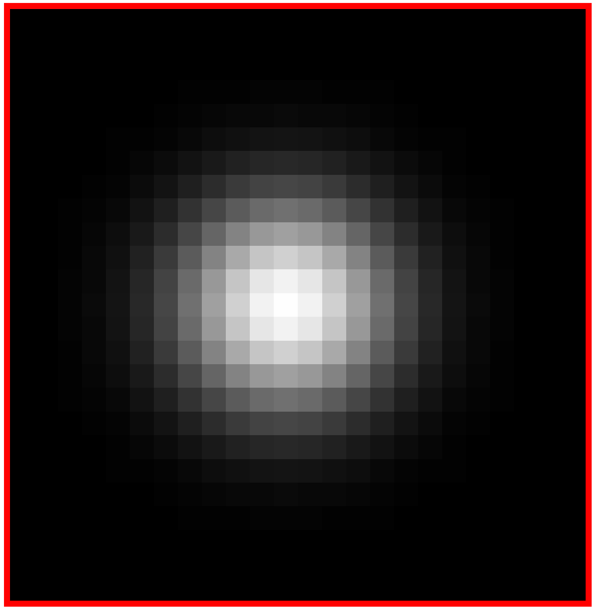
$$y[n_1, n_2] = x[n_1, n_2] * h[n_1, n_2]$$

- this is of great practical importance:
  - for an LSI system the response to any input can be obtained by the convolution with this impulse response
  - the IR fully characterizes the system
  - it is all that I need to measure

2D convolution

- note that the impulse response is really just that
  - e.g. suppose that I want to recover the blurring function of a bad camera
  - all I have to do is shine a spot of light at it
  - this is the impulse response of the camera

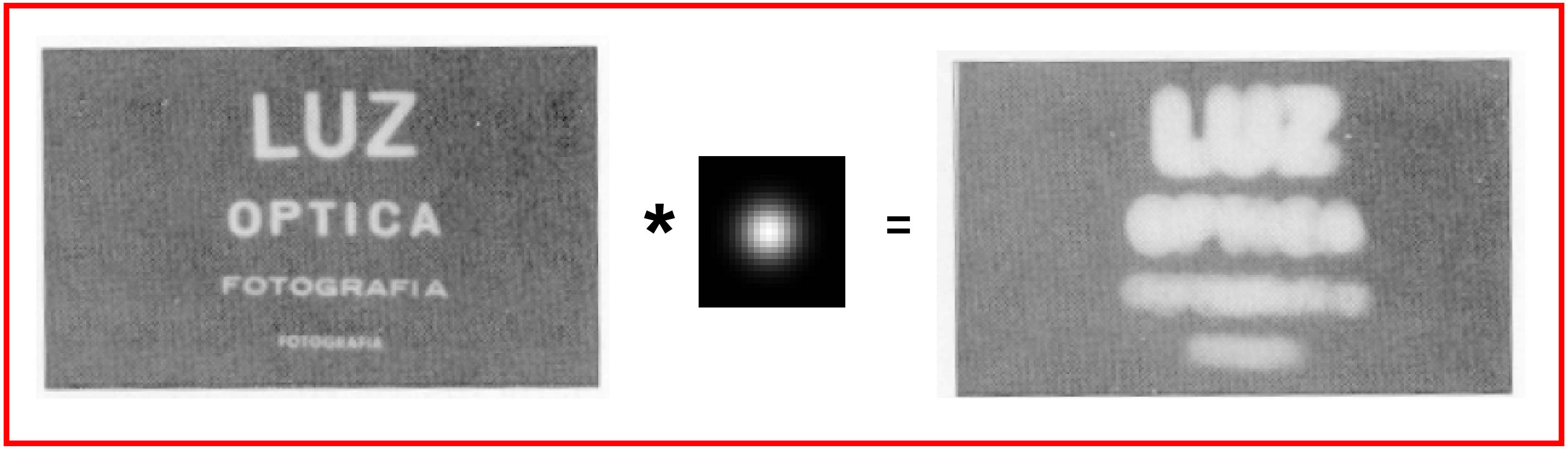




- what is the camera response to any image?
  - well, I just need to convolve the image with this impulse response

2D convolution

- you can try this at home



- the fact that this works for any image is the "miracle" of LSI systems!

Any questions?