ECE-161C
Color

Nuno Vasconcelos

ECE Department, UCSD

(with thanks to David Forsyth)
Image formation

we are studying the process of image formation

two questions

• what 3D point projects into pixel (x,y)?
• what is the light incident on the pixel?

determine the image intensity at the pixel

incident light

reflected light

image brightness
Geometry

- **geometry** answers the first question

- **pinhole camera:**
  - point \((x,y,z)\) in 3D scene projected into image pixel of coordinates \((x', y')\)
  - according to the **perspective projection equation**:

\[
\begin{pmatrix}
  x' \\
  y' \\
  1
\end{pmatrix} = \begin{bmatrix}
  f \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix} \begin{pmatrix}
  x \\
  y \\
  z \\
  1
\end{pmatrix}
\]
Lambertian surfaces

- radiometry answers the second
  - if surface is Lambertian
  - if $n$ is the surface normal, $s$ the light direction, $\rho$ the surface albedo, and $E$ the source power

$$P(P_i) = E \rho_a(P_0) \vec{n}(P_0) \cdot \vec{s}$$
Color

how do we perceive color?

color perception is the result of evolution

let’s look at the human eye

- light enters through pupil
- this has the role of a lens
- projects into the retina
- this is the “camera plane”
- retina covered by receptors that transform light into electric pulses
- sent to the brain through the optic nerve
Color

- there are only two types of receptors:
  - rods:
    - responsible for vision at low light levels, do not mediate color vision
  - cones:
    - active at higher light levels, capable of color vision
- density:
  - rods dominate in the periphery of the eye,
  - the center is mostly composed of cones
- three types of cones:
  - denoted by S, M, and L
Rod vs cone density

- very high concentration of cones in the center
- high resolution and color
- density decays very quickly

center  periphery
Color receptors

- The three types of cones, vary in their sensitivity to light at different wavelengths.
- Picture shows a cone mosaic, spatial distribution of the different types of cones.
- Blue: S cones
  - Red: L cones
  - Green: M cones
- Note the different densities.
Color receptors

- relative sensitivity as a function of wavelength. S (for short) cone responds most strongly at short wavelengths; the M (for medium) at medium wavelengths and the L (for long) at long wavelengths.

- occasionally called B, G and R cones respectively, but that’s misleading - you don’t see red because your R cone is activated.
Radiometry for color

- the three types of cones are filters, specialized to certain wavelengths
- why does this happen?
- because:
  - the visible spectrum covers a range of wavelengths
  - radiance is a function of wavelength $\lambda$
  - e.g. if you put a color filter in front of your lens, only certain wavelengths go through
  - and all objects reflect light differently at different wavelengths
  - this is what makes them look like they have color
important:

• albedo is a function of wavelength

\[ \rho(P, \lambda) \]

• example of how it varies with wavelength for different types of leaves (measurements by E.Koivisto).
Spectral albedo

how does this change our radiometry equation?

- $n$ (surface normal), $s$ (light direction), do not change with wavelength
- the dependence on wavelength can come from $\rho$ the surface albedo, or $E$ the source power

\[
P(P_i, \lambda) = E(\lambda) \rho_a(P_0, \lambda) \vec{n}(P_0) \cdot \vec{s}
\]
Spectral albedo

- light
  - can play a significant role in certain environments
  - this is why your perception of color is unreliable in night clubs
- for “white light”, $E$ is constant
- all dependence is due to the albedo

$$P(P_i, \lambda) = E \rho_a(P_0, \lambda) \vec{n}(P_0).\vec{s}$$

- e.g. different threads have different albedo
- color changes from location to location
The appearance of colors

Principle of trichromacy:

- it is possible to match almost all colors, using only three primary sources.
- this is not surprising given that we have three types of receptors
- and these are filters for some wavelengths
- anything that falls out of the combination of the three primaries can’t be seen by us
Color matching experiments

- Subject shown a **split field**:
  - one side shows the light whose color one wants to measure,
  - other a weighted mixture of primaries (fixed lights).

Subject adjusts dials so as to make mixture equal to test
Color matching experiments (cont’d)

- most colors can be represented as a mixture of $P_1, P_2, P_3$

$$M(\lambda) = a P_1(\lambda) + b P_2(\lambda) + c P_3(\lambda)$$

where the = sign should be read as “matches”

- this is additive matching.

- important because if two people who agree on $P_1, P_2, P_3$
  need only supply $(a, b, c)$ to describe a color.

- some colors can’t be matched like this: instead we need

$$M(\lambda) + a P_1(\lambda) = b P_2(\lambda) + c P_3(\lambda)$$

- this is subtractive matching.

- we can interpret it as $(-a, b, c)$
The principle of trichromacy

Experimental facts:

- three primaries will work for most people if we allow subtractive matching
  - Exceptional people can match with two or only one primary.
  - This could be caused by a variety of deficiencies.
- most people make the same matches.
  - There are some anomalous trichromats, who use three primaries but make different combinations to match.
Grassman’s Laws

- **important property:** color matching is linear

1. mixture of coordinates matches the mixture of the lights

   \[
   T_1 = a_1 P_1 + b_1 P_2 + c_1 P_3 \\
   T_2 = a_2 P_1 + b_2 P_2 + c_2 P_3
   \]

   \[
   T_1 + T_2 = (a_1 + a_2)P_1 + (b_1 + b_2)P_2 + (c_1 + c_2)P_3
   \]

2. equal coordinates means equal lights

   \[
   T_1 = aP_1 + bP_2 + cP_3 \\
   T_2 = aP_1 + bP_2 + cP_3
   \]

   \[
   T_1 = T_2
   \]

3. matching is linear

   \[
   T_1 = aP_1 + bP_2 + cP_3 \iff kT_1 = kaP_1 + kbP_2 + kcP_3
   \]

- these are known as Grassman’s Laws
Linear color spaces

- because color matching is linear in the primaries it makes sense to think of the primaries as the basis of a linear color space

- note that the space is infinite dimensional
  1. think of $L(\lambda)$ as $L(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  2. take $N$ to infinity

- the space of valid colors is a 3D-subspace

- problem: the basis is not necessarily orthogonal
Basis functions

➤ you are probably used to vector spaces with a finite number of components
  • $f = (f_1, \ldots, f_n)$

➤ when $n$ goes to infinity we have a function
  • $f = f(t)$

➤ this is also a vector space, but now a vector space of functions
  • everything that you have learned still holds, we only need to change our operators a little
  • the main difference is that summations become integrations
  • e.g. for the dot product between $f$ and $g$

\[
\langle f, g \rangle = \sum_i f_i g_i \quad \Rightarrow \quad \langle f, g \rangle = \int f(t) g(t) dt
\]
Basis functions

A basis for a subspace of dimension $d$ is a set of $d$ functions $b_1(t), \ldots, b_d(t)$ such that

- any function in the space is a linear combination of these $d$ functions (the $b_i(t)$ span the space)
  
  $$f(t) = \sum_{i=1}^{d} \alpha_i b_i(t)$$
  
- the $b_i(t)$ are linearly independent (i.e. there is no linear combination that will add to the zero function other than $\alpha_i=0$)

The basis is orthonormal if the $b_i(t)$ are orthogonal functions and have unit norm

$$\langle b_i(t), b_j(t) \rangle = \int b_i(t)b_j(t)dt = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$
Fourier series

Note that you have already seen this.

For example, the Fourier series is the result of the projection of a function $f(x)$ on the basis whose functions are $\cos(nx)$ and $\sin(nx)$.

Note that the basis is orthogonal, since

$$
\int \cos(nx) \sin(mx) \, dx = 0 \quad \int \cos(nx) \cos(mx) \, dx = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases}
$$

$$
\int \sin(nx) \sin(mx) \, dx = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases}
$$

The series coefficients are the coordinate on this basis:

$$
\alpha_0 = \frac{1}{\pi} \int f(x) \, dx \quad \alpha_n = \frac{1}{\pi} \int f(x) \cos(nx) \, dx \quad \beta_n = \frac{1}{\pi} \int f(x) \sin(nx) \, dx
$$
Fourier series

- The function is a linear combination of the basis functions

\[ f(x) = \frac{1}{2} \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \sum_{n=0}^{\infty} \beta_n \sin(nx) \]

- Note that the basis functions do not have unit norm, and so we need the $\frac{1}{\pi}$ factors

- The picture is
Linear color spaces

- back to color
- because color matching is linear in the primaries it makes sense to think of the primaries as the basis of a linear color space
- note that the space is infinite dimensional
- the space of valid colors is a 3D-subspace
- problem: the basis is not necessarily orthogonal
Color matching functions

- how do we get the color coordinates?
  - pick a source $\delta(\lambda-\lambda_0)$ of unit radiance at wavelength $\lambda$
  - we know that there is a set of weights that matches it
  - denote the weight of primary $P_i$ by $f_i(\lambda_0)$

\[
\delta(\lambda-\lambda_0) = \sum_{i=1}^{3} f_i(\lambda_0) P_i(\lambda)
\]

- repeat for all $\lambda_0$

- the $f_i(\lambda)$ are called color matching functions because we can write

\[
L(\lambda) = \int L(\lambda_0) \delta(\lambda_0 - \lambda) d\lambda_0
= \sum_i \left\{ \int L(\lambda_0) f_i(\lambda_0) d\lambda_0 \right\} P_i(\lambda)
= \sum \omega_i P_i(\lambda)
\]
the color coordinates are the projections of the spectral radiance on the three matching functions

\[
a = \int L(\lambda_0) f_1(\lambda_0) d\lambda_0 \\
b = \int L(\lambda_0) f_2(\lambda_0) d\lambda_0 \\
c = \int L(\lambda_0) f_3(\lambda_0) d\lambda_0
\]

note that this means that we can specify the color space by specifying a set of matching functions

this can sometimes lead to primaries that are not physically feasible (e.g. if we constrain the matching functions to be non-negative)

OK because we really only care about the coordinates
In summary

color can be represented in **different color spaces**

- a color space is defined by a set of three primaries
  \[ \{P_0(\lambda), P_1(\lambda), P_2(\lambda)\} \]

- any color is a linear combination of these
  \[ L(\lambda) = aP_0(\lambda) + bP_1(\lambda) + cP_2(\lambda) \]

- the coordinates are found by projection onto the matching functions
  \[ a = \int L(\lambda_0) f_1(\lambda_0) d\lambda_0; \quad b = \int L(\lambda_0) f_2(\lambda_0) d\lambda_0 \]
  \[ c = \int L(\lambda_0) f_3(\lambda_0) d\lambda_0 \]

- the matching functions are the solutions of
  \[ \delta(\lambda - \lambda_0) = \sum_{i=1}^{3} f_i(\lambda_0) P_i(\lambda) \]
In summary

why do we need different color spaces?

• many reasons
• some applications require a representation that matches human color perception
• others need color primaries that are convenient for certain tasks, such as building and LCD or a CRT

in general

• we are given the color space, and only work with components \((a,b,c)\)
• color is a three dimensional vector
• we have three “images” known as the “three color channels”
• once in a while, we need to convert between color spaces
in summary

► for example, images are commonly represented in the RGB space
► each pixel is vector of three numbers
► these are the amounts of Red, Green, and Blue
► what exactly is the RGB space?
  • we can look at the associated matching functions
Linear color spaces

- **RGB**: primaries are monochromatic energies at 645.2nm, 526.3nm, 444.4nm.
- Color matching functions have negative parts.
- Some colors can be matched only subtractively.
RGB space

- primaries are the phosphors monitors use as primaries
- the color space is a cube
- axes represent the amount of red, green, and blue
- each color has coordinates between 0 and 1 (0 and 255)
Linear color spaces

- **CIE XYZ**: color matching functions positive
- Due to this primaries are imaginary, but have convenient properties
- Color coordinates are (X,Y,Z), where X is amount of X primary, etc.
CIE xy

- 2D is easier to visualize than 3D
- usually work with CIE xy, where
  - $x = X/(X+Y+Z)$
  - $y = Y/(X+Y+Z)$
- this is the intersection of the color space with the plane $X+Y+Z=1$
Perceptual color spaces

- human perception of color is best described in terms of three fundamental color properties
- **hue**: this is what you would refer to as the color itself
  
  e.g. red vs yellow

- note that there is a **circular structure** (we start and finish with red)

- for this reason color is usually visualized in a **color wheel**
Color perception

- **saturation**: this is the adjective: e.g. “vivid” red, “pale” yellow
- it is the radial coordinate on the color wheel
- colors at the center are unsaturated, colors at the boundaries are highly saturated
Color perception

- **intensity**: is the amount of brightness
- “dark” yellow vs “light” yellow
- the color wheel can be replicated at each intensity level

Note that:
- the center is always the intensity level
- at zero intensity, hue and saturation are irrelevant
- saturation becomes more important at higher intensities
Perceptual color spaces

- **HSV**: hue, saturation, value (intensity) is a natural “perceptual” color space
- very non-linear, colors form a cone
- this can be bad for some applications
- e.g. TVs where primaries have nothing to do with perception but with building CRTs
- but very good for others
Perceptual color spaces

- note that, in general, we can extract the three properties from any space
- this picture shows H, S, and V in CIE xy
- the problem is that the transformation is non-linear
- in CIExy two colors that are near do not necessarily “look” similar
MacAdam ellipses

- ellipses: colors that human observers matched to color on their center; boundary shows just noticeable difference.
- left: magnified 10x. The ellipses at the top are larger than those at the bottom, and rotate as they move up.
- difference in x,y is poor guide to color difference!
Perceptual color spaces

- **Uniform**: equal (small!) steps give the same perceived color changes

- this is important, for example for image retrieval based on color similarity

- compute an histogram of color values for each image and measure image similarity by the similarity of the color histograms

- works surprisingly well (more on this later)
Conversions

▶ in practice, we often have to do color-space conversions

▶ here are some of the formulas (for more details see http://www.easyrgb.com/math.html)

- RGB to XYZ: see homework
- CIE XYZ to CIE u’v’

\[
(u', v') = \left( \frac{4X}{X + 15Y + 3Z}, \frac{9Y}{X + 15Y + 3Z} \right)
\]

- CIE XYZ to L*a*b

\[
L^* = 116 \left( \frac{Y}{Y_n} \right)^{\frac{1}{3}} - 16 \quad a^* = 500 \left[ \left( \frac{X}{X_n} \right)^{\frac{1}{3}} - \left( \frac{Y}{Y_n} \right)^{\frac{1}{3}} \right] \quad b^* = 200 \left[ \left( \frac{Y}{Y_n} \right)^{\frac{1}{3}} - \left( \frac{Z}{Z_n} \right)^{\frac{1}{3}} \right]
\]

\((X_n, Y_n, Z_n)\) are \((X, Y, Z)\) coordinates of a reference white patch.
Conversions

- **RGB to CMY**
  \[ C = 1-R \quad M = 1-G \quad Y=1-B \]

- **CMY to CMYK**
  \[ K = \min(C,M,Y) \quad C = C-K \quad M = M-K \quad Y=Y-K \]

- **RGB to YCrCb**
  \[ Y = 0.29900R + 0.58700G + 0.11400B \]
  \[ C_b = -0.16874R - 0.33126G + 0.50000B \]
  \[ C_r = 0.50000R-0.41869G – 0.08131B \]
Conversions

• **RGB to HSV**

\[ Min = \min(R,G,B); \quad Max = \max(R,G,B); \]

\[ \Delta = Max - Min; \quad V = Max; \]

\[
\text{if ( } \Delta \text{ == 0 ) } \{ \textbf{H} = 0; \textbf{S} = 0 \}\]
\[
\text{else } \{ \]
\[
\textbf{S} = \Delta / Max; \]

\[
\Delta R = \left( [(Max - R) / 6 + (\Delta / 2)] / \Delta; \right.
\]
\[
\Delta G = \left( [(Max - G) / 6 + (\Delta / 2)] / \Delta \right.
\]
\[
\Delta B = \left( [(Max - B) / 6 + (\Delta / 2)] / \Delta \right)
\]

\[
\text{if ( } R \text{ == Max ) } \textbf{H} = \Delta B - \Delta G
\]
\[
\text{else if ( } G \text{ == Max ) } \textbf{H} = ( 1 / 3 ) + \Delta R - \Delta B
\]
\[
\text{else if ( } B \text{ == Max ) } \textbf{H} = ( 2 / 3 ) + \Delta G - \Delta R
\]
\[
\text{if ( } \textbf{H} \text{ < 0 ) ; } \textbf{H} += 1; \text{ if ( } \textbf{H} \text{ > 1 ) ; } \textbf{H} -= 1
\}
Any questions?