# Discrete Fourier Transform

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# The Discrete-Space Fourier Transform

- as in 1D, an important concept in linear system analysis is that of the Fourier transform
- the Discrete-Space Fourier Transform is the 2D extension of the Discrete-Time Fourier Transform

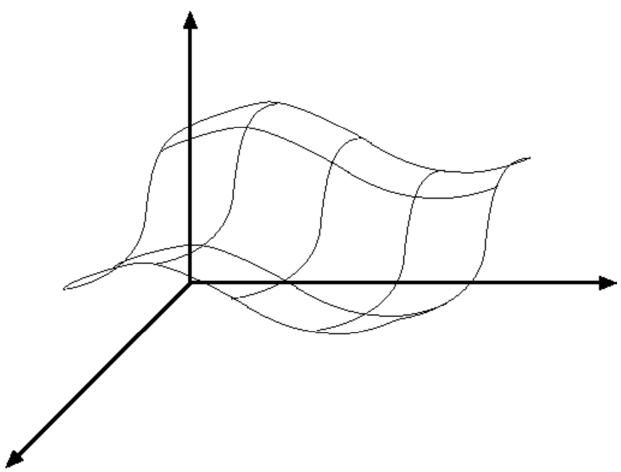
$$\boxed{\begin{aligned} X(\omega_1,\omega_2) &= \sum_{n_1}\sum_{n_2} x[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \\ x[n_1,n_2] &= \frac{1}{(2\pi)^2}\iint X(\omega_1,\omega_2)e^{j\omega_1 n_1}e^{j\omega_2 n_2}d\omega_1 d\omega_2 \end{aligned}}$$

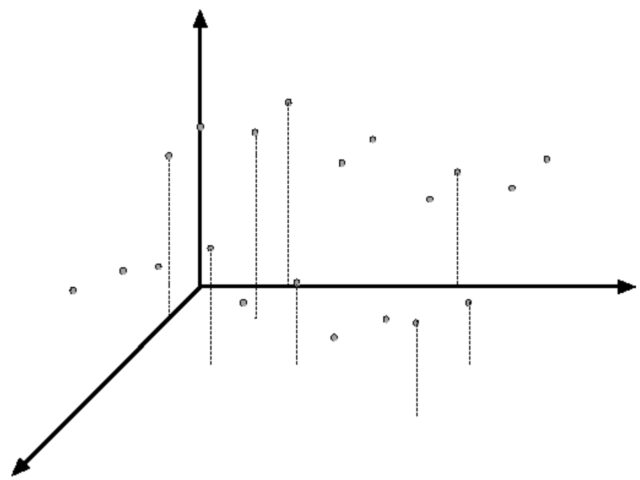
- note that this is a continuous function of frequency
  - inconvenient to evaluate numerically in DSP hardware
  - we need a discrete version
  - this is the 2D Discrete Fourier Transform (2D-DFT)
- before that we consider the sampling problem

# Sampling in 2D

- consider an analog signal $x_c(t_1,t_2)$ and let its analog Fourier transform be $X_c(\Omega_1,\Omega_2)$
  - we use capital $\Omega$ to emphasize that this is analog frequency
- sample with period $(T_1,T_2)$ to obtain a discrete-space signal

$$\boxed{x[n_1,n_2] = x_c(t_1,t_2)\Big|_{t_1=n_1T_1;\,t_2=n_2T_2}}$$





# Sampling in 2D

- relationship between the Discrete-Space FT of $x[n_1,n_2]$ and the FT of $x_c(t_1,t_2)$ is simple extension of 1D result

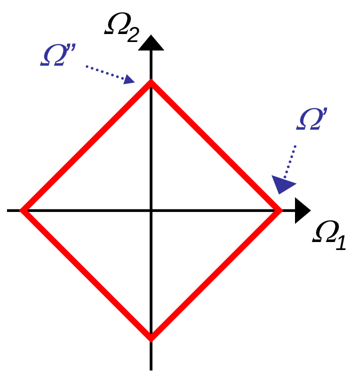
$$\boxed{X(\omega_1,\omega_2)=\frac{1}{T_1T_2}\sum_{r_1=-\infty}^{\infty}\sum_{r_2=-\infty}^{\infty}X_c\left(\frac{\omega_1-2\pi r_1}{T_1},\frac{\omega_2-2\pi r_2}{T_2}\right)}$$

DSFT of $x[n_1,n_2]$
"discrete spectrum"

FT of $x_c(\omega_1,\omega_2)$
"analog spectrum"

- Discrete Space spectrum is sum of replicas of analog spectrum
  - in the "base replica" the analog frequency $\Omega_1$ ($\Omega_2$) is mapped into the digital frequency $\Omega_1 T_1$ ($\Omega_2 T_2$)
  - discrete spectrum has periodicity $(2\pi,2\pi)$

# For example



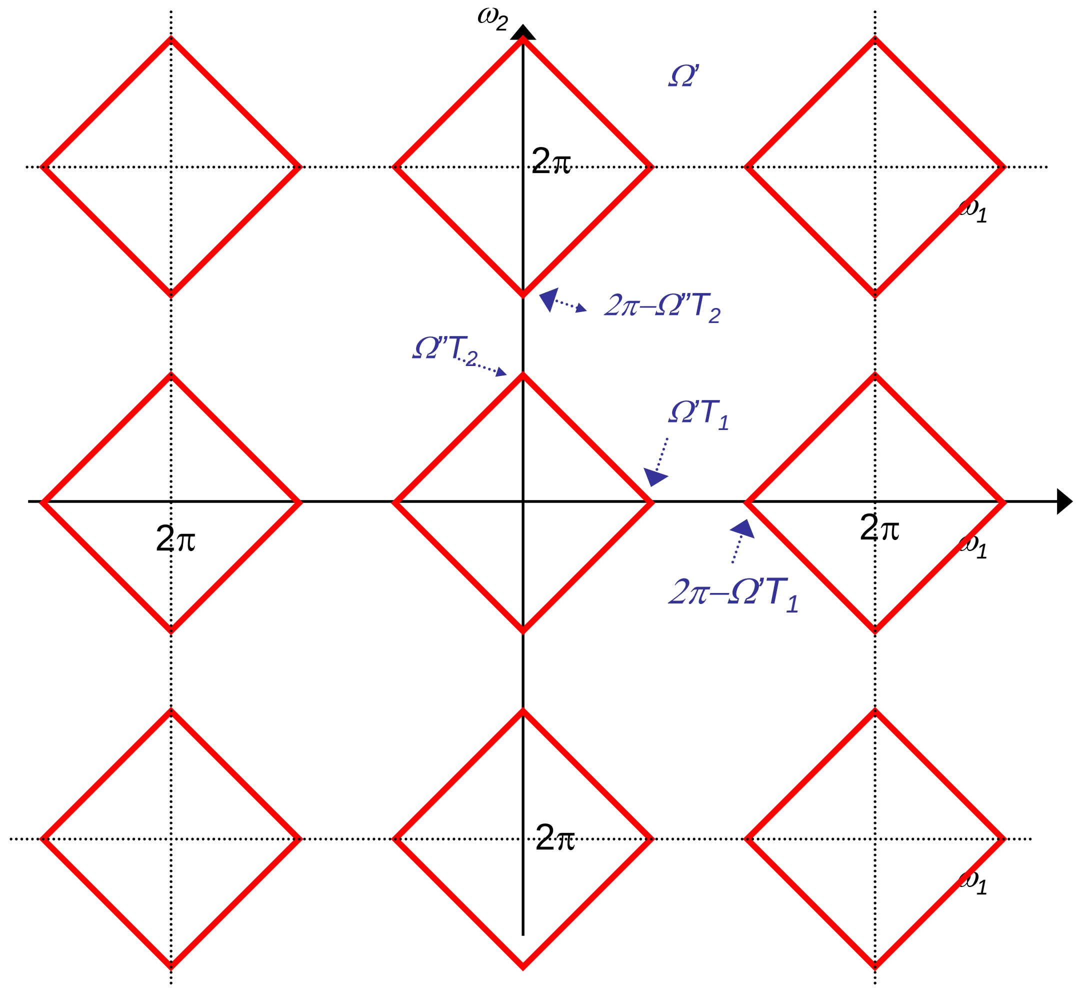

$$\Omega' \rightarrow \alpha = \Omega' T_1$$

$$\Omega'' \rightarrow \beta = \Omega'' T_2$$

- no aliasing if

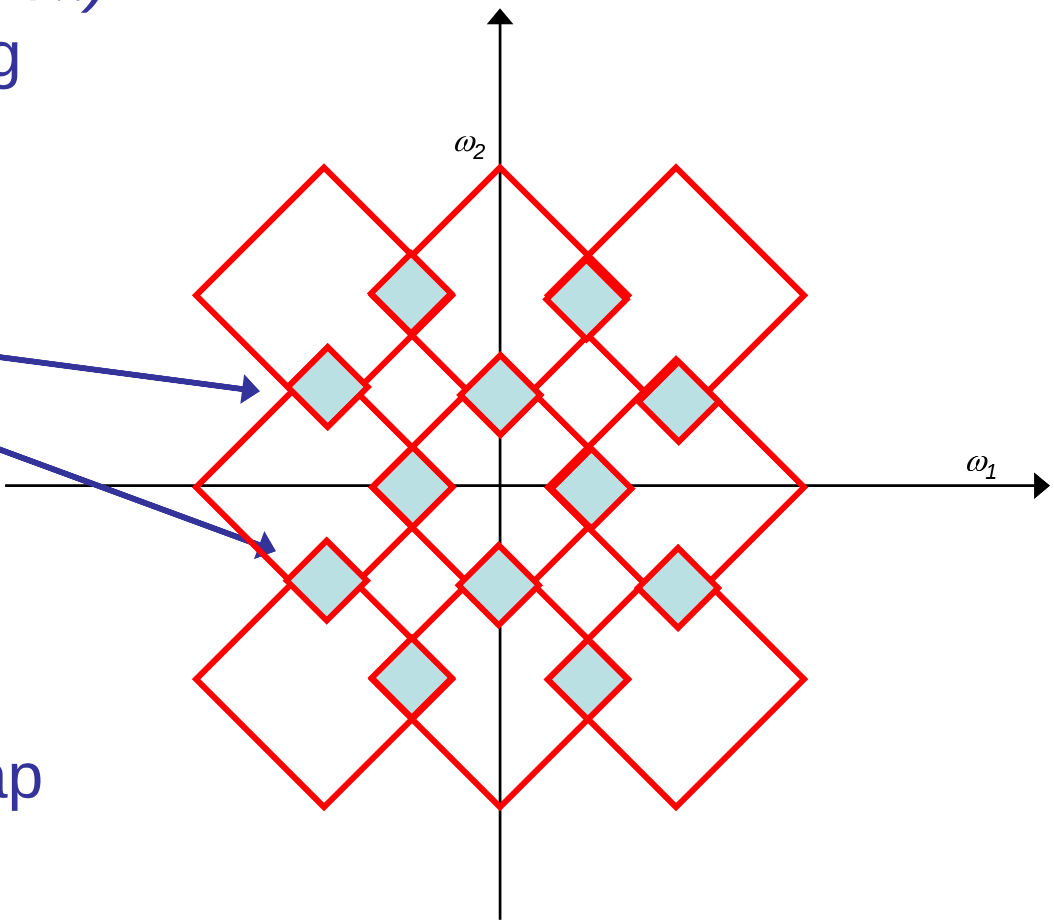
$$\begin{cases} \Omega' T_1 \le 2\pi - \Omega' T_1 \\ \Omega'' T_2 \le 2\pi - \Omega' T_2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\begin{cases} T_1 \le \pi / \Omega' \\ T_2 \le \pi / \Omega'' \end{cases}}$$



# Aliasing

- the frequency $(\Omega'/\pi, \Omega''/\pi)$ is the critical sampling frequency
- below it we have aliasing




- this is just like the 1D case, but now there are more possibilities for overlap

# Reconstruction

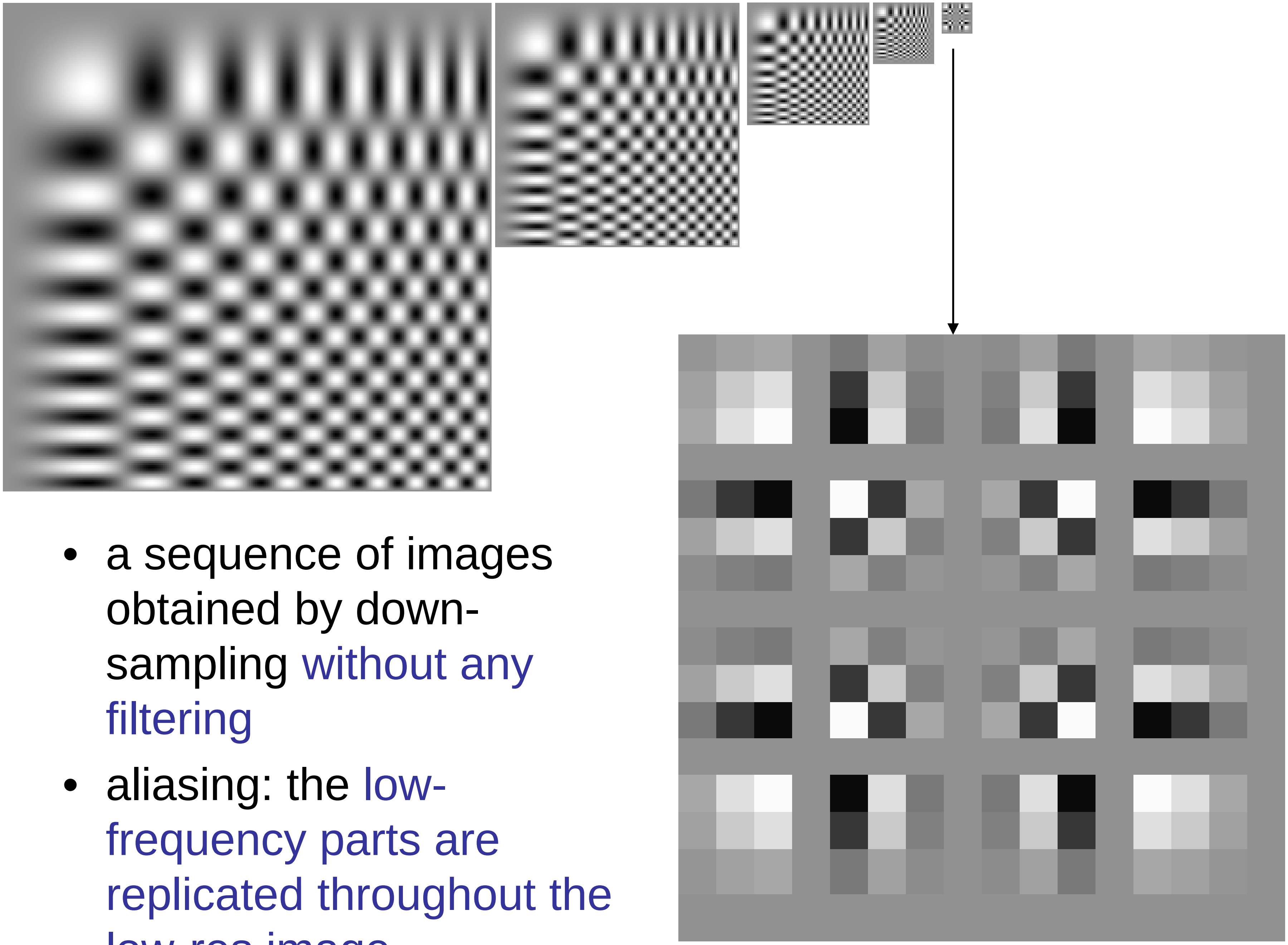
- if there is no aliasing we can recover the signal in a way similar to the 1D case

$$y_c(t_1,t_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2]\frac{\sin\frac{\pi}{T_1}(t_1-n_1T_1)}{\frac{\pi}{T_1}(t_1-n_1T_1)}\frac{\sin\frac{\pi}{T_2}(t_2-n_2T_2)}{\frac{\pi}{T_2}(t_2-n_2T_2)}$$

- note: in 2D there are many more possibilities than in 1D
  - e.g. the sampling grid does not have to be rectangular, e.g. hexagonal sampling when $T_2 = T_1/sqrt(3)$ and

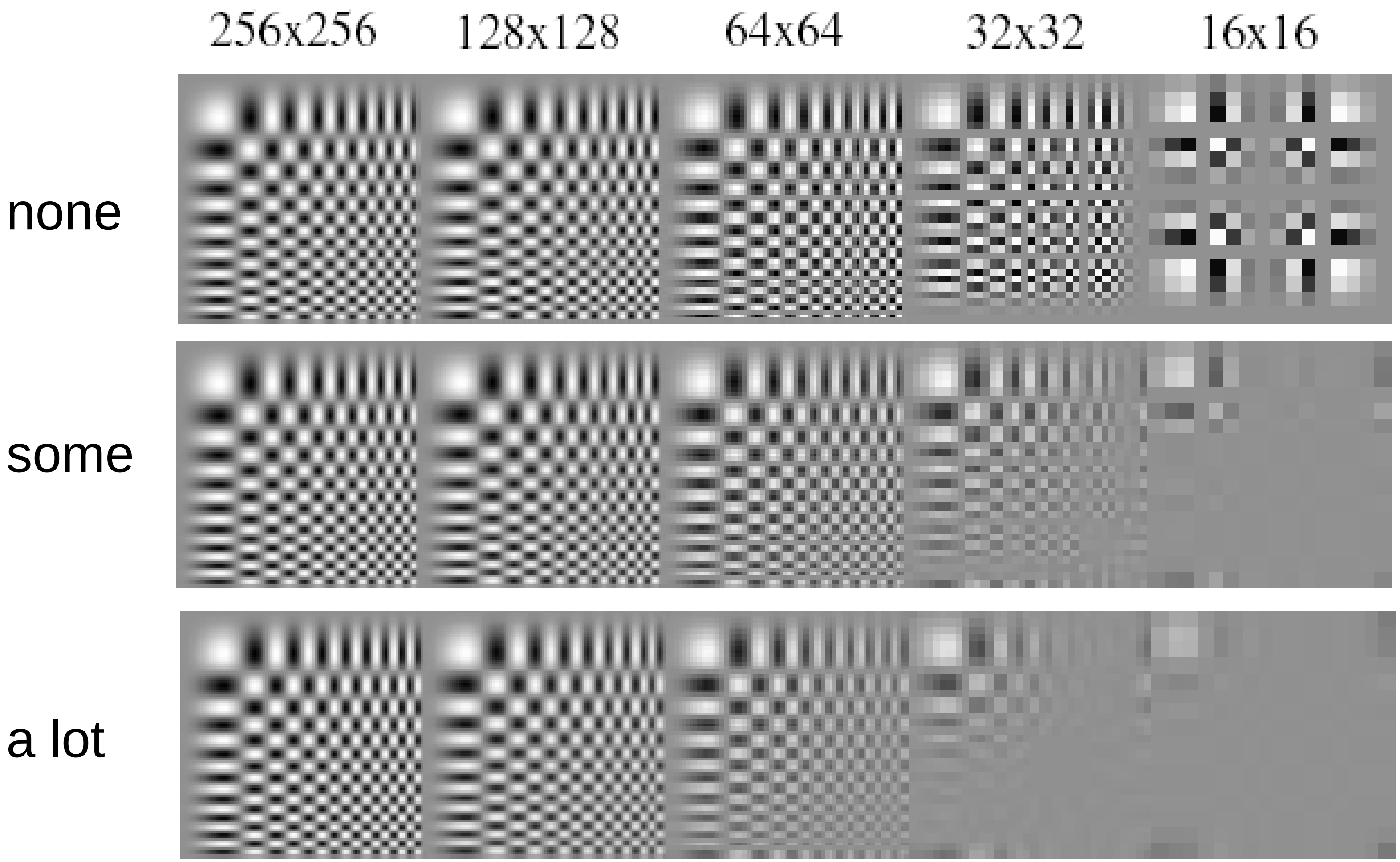
$$x[n_1,n_2]=\begin{cases} x_c(t_1,t_2)\big|_{t_1=n_1T_1;t_2=n_2T_2} & n_1,n_2 \text{ both even or odd} \\ 0 & \text{otherwise}\end{cases}$$

  - in practice, however, one usually adopts the rectangular grid



- a sequence of images obtained by down-sampling without any filtering
- aliasing: the low-frequency parts are replicated throughout the low-res image

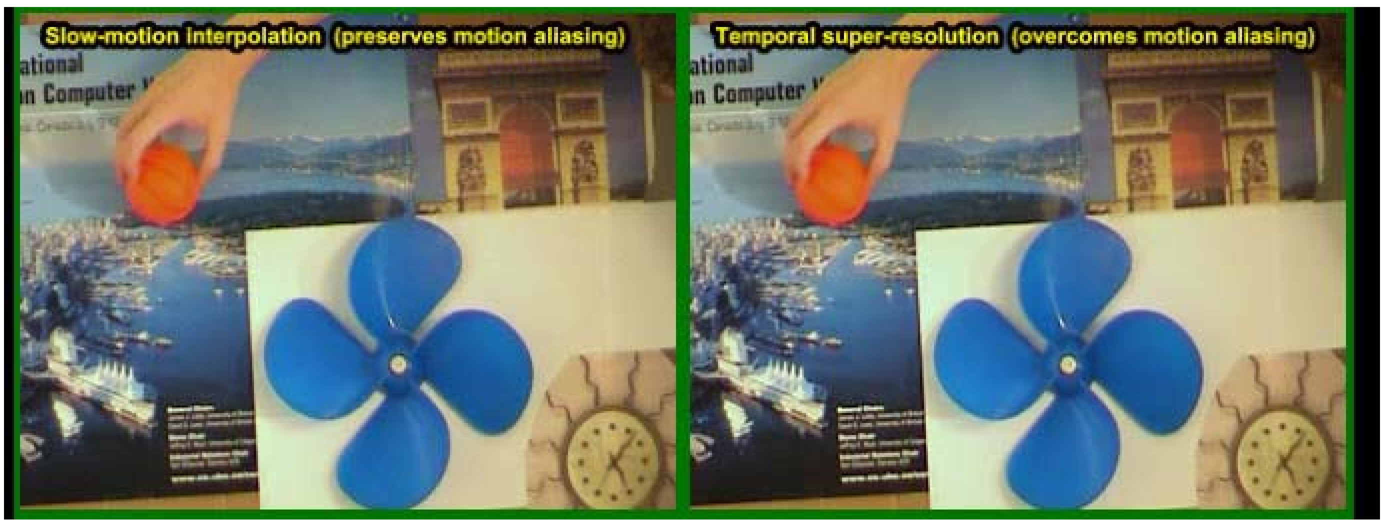
# The role of smoothing




- too little leads to aliasing
- too much leads to loss of information

# Aliasing in video

- video frames are the result of temporal sampling
  - fast moving objects are above the critical frequency
  - above a certain speed they are aliased and appear to move backwards
  - this was common in old western movies and become known as the “wagon wheel” effect
  - here is an example: super-resolution increases the frame rate and eliminates aliasing




from **“Space-Time Resolution in Video” by E. Shechtman, Y. Caspi and M. Irani**

**(PAMI 2005).**

# 2D-DFT

- the 2D-DFT is obtained by sampling the DSFT at regular frequency intervals

$$\boxed{X[k_1,k_2] = X(\omega_1,\omega_2)\Big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2}}$$

- this turns out to make the 2D-DFT somewhat harder to work with than the DSFT
  - it is the same as in 1D
  - you might remember that the inverse transform of the product of two DFTs is not the convolution of the associated signals
  - but, instead, the "circular convolution"
  - where does this come from?
- it is better understood by first considering the 2D Discrete Fourier Series (2D-DFS)

# 2D-DFS

- it is the natural representation for a periodic sequence
- a sequence $\underline{x}[n_1,n_2]$ is periodic of period $N_1 x N_2$ if

$$\boxed{\begin{aligned}\underline{x}[n_1,n_2] &= \underline{x}[n_1+N_1,n_2] \\ &= \underline{x}[n_1,n_2+N_2], \quad \forall n_1,n_2\end{aligned}}$$

- note that

$$\boxed{\underline{X}(r_1,r_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2]r_1^{-jn_1}r_2^{-jn_2}}$$

- makes no sense for a periodic signal
  - the sum will be infinite for any pair $r_1,r_2$
  - neither the 2D DSFT or the Z-transform will work here

# 2D-DFS

- the 2D-DFS solves this problem
- it is based on the observation that
  - any periodic sequence can be represented as a weighted sum of complex exponentials of the form

$$\boxed{\underline{X}(k_1, k_2) \times e^{j\frac{2\pi}{N_1}k_1 n_1} \times e^{j\frac{2\pi}{N_2}k_2 n_2}, \quad 0 \le k_1 \le N_1 - 1, \quad 0 \le k_2 \le N_2 - 1}$$

  - this is a simple consequence of the fact that

$$\boxed{e^{j\frac{2\pi}{N_1}k_1 n_1} \times e^{j\frac{2\pi}{N_2}k_2 n_2}, \quad 0 \le k_1 \le N_1 - 1, 0 \le k_2 \le N_2 - 1}$$

  - is an orthonormal basis of the space of periodic sequences

# 2D-DFS

- the 2D-DFS relates $\underline{x}[n_1,n_2]$ and $\underline{X}[k_1,k_2]$

$$\boxed{\begin{aligned}\underline{X}(k_1,k_2) &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\underline{x}[n_1,n_2]\,e^{-j\frac{2\pi}{N_1}k_1n_1}e^{-j\frac{2\pi}{N_2}k_2n_2} \\ \underline{x}[n_1,n_2] &= \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\underline{X}[k_1,k_2]\,e^{j\frac{2\pi}{N_1}k_1n_1}e^{j\frac{2\pi}{N_2}k_2n_2}\end{aligned}}$$

- note that $\underline{X}[k_1,k_2]$ is also periodic outside

$$\boxed{0 \le k_1 \le N_1 - 1, \quad 0 \le k_2 \le N_2 - 1}$$

- like the DSFT,
  - properties of the 2D-DFS are identical to those of the 1D-DFS
  - with the straightforward extension of separability

# Periodic convolution

- like the Fourier transform,
  - the inverse transform of multiplication is convolution

$$\boxed{\underline{x}[n_1, n_2] * \underline{x}[n_1, n_2] \quad \overset{DFS}{\longleftrightarrow} \quad \underline{X}(k_1, k_2) \times \underline{Y}(k_1, k_2)}$$

  - however, we have to be careful about how we define convolution
  - since the sequences have no end, the standard definition

$$\boxed{y[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$

    makes no sense
  - e.g. if x and h are both positive sequences, this will allways be infinite

# Periodic convolution

- to deal with this, we introduce the idea of periodic convolution
- instead of the regular definition

$$\boxed{x * y = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$

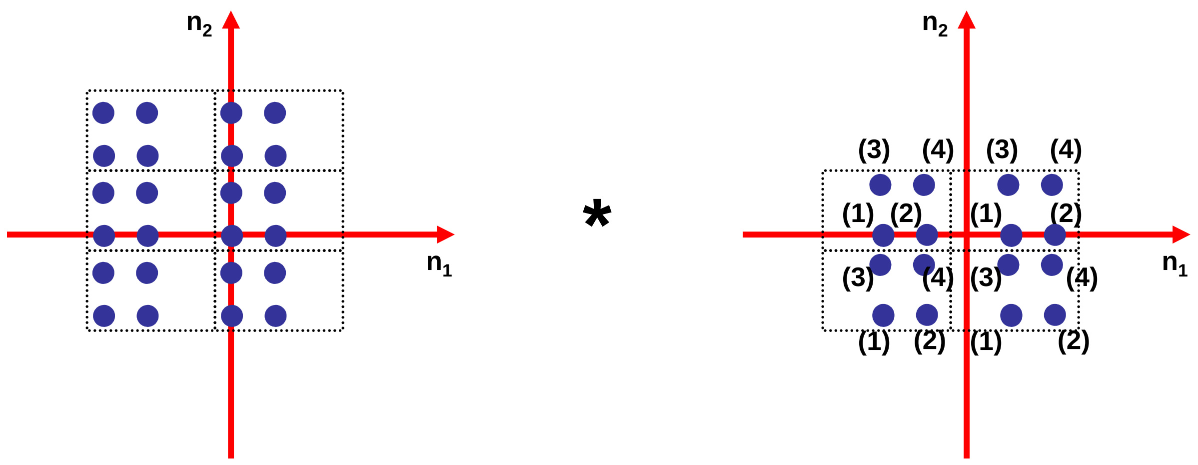
- which, from now on, we refer to as linear convolution
- periodic convolution only considers one period of our sequences

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$

- the only difference is in the summation limits

# Periodic convolution

- this is simple, but produces a convolution which is substantially different
- let's go back to our example, now assuming that the sequences have period ($N_1$=3,$N_2$=2)



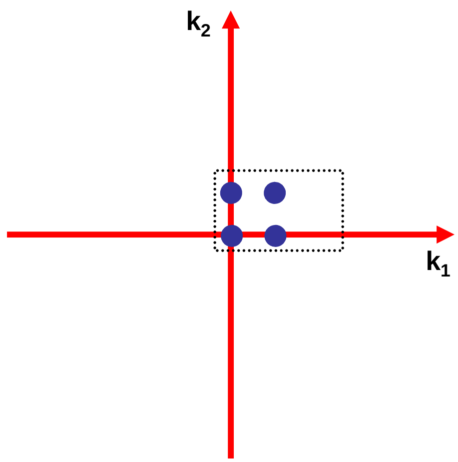

$\underline{x}[n_1, n_2]$ $\underline{h}[n_1, n_2]$

- as before, we need four steps

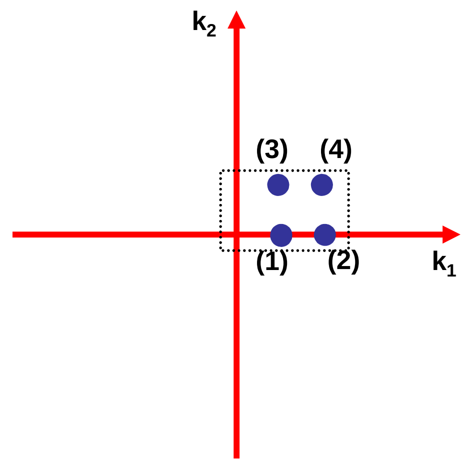
# Periodic convolution

- **step 1):** express sequences in terms of *($k_1$, $k_2$), and consider one period only*

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$




$x[k_1, k_2]$



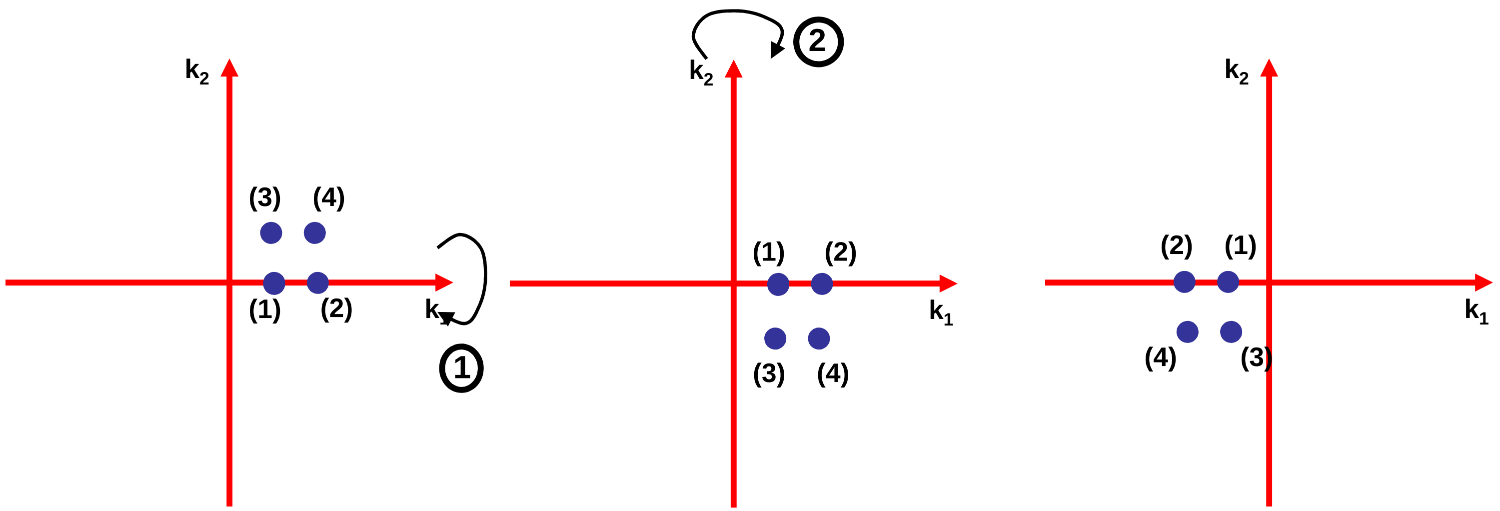

$h[k_1, k_2]$

we next proceed exactly as before

# Periodic convolution

- **step 2):** invert $h(k_1, k_2)$

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$

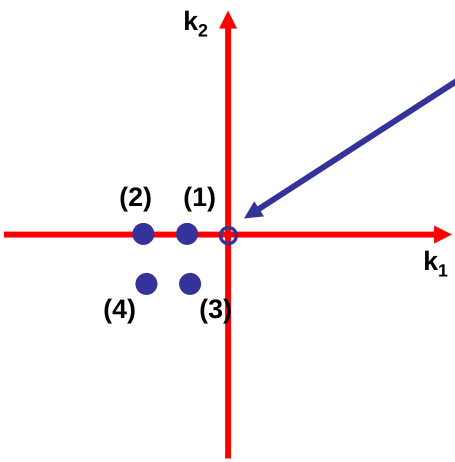



$h[k_1, k_2]$ $\qquad$ $h[k_1, -k_2]$ $\qquad$ $g[k_1, k_2] = h[-k_1, -k_2]$

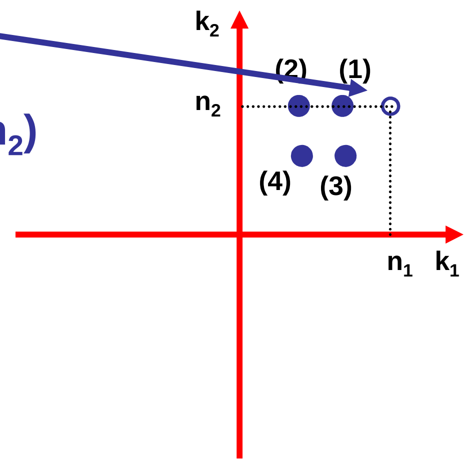
# Periodic convolution

- **step 3):** shift $g(k_1, k_2)$ by $(n_1, n_2)$

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$




this sends whatever is at (0,0) to $(n_1,n_2)$

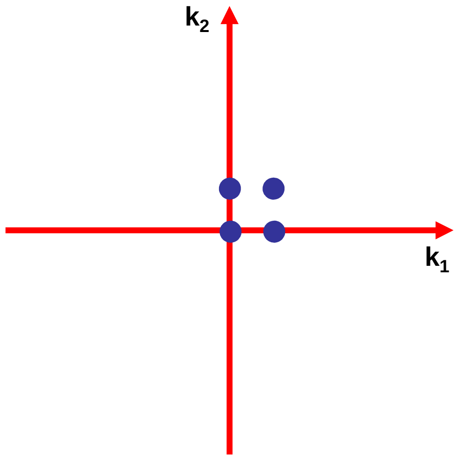


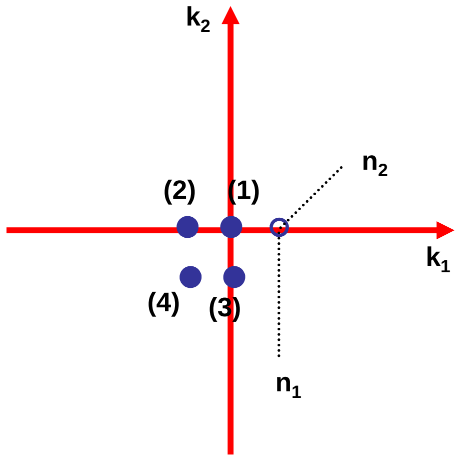

$$g[k_1, k_2] = h[-k_1, -k_2]$$

$$g[k_1 - n_1, k_2 - n_2] = h[n_1 - k_1, n_2 - k_2]$$

# Periodic convolution

- e.g. for $(n_1,n_2) = (1,0)$



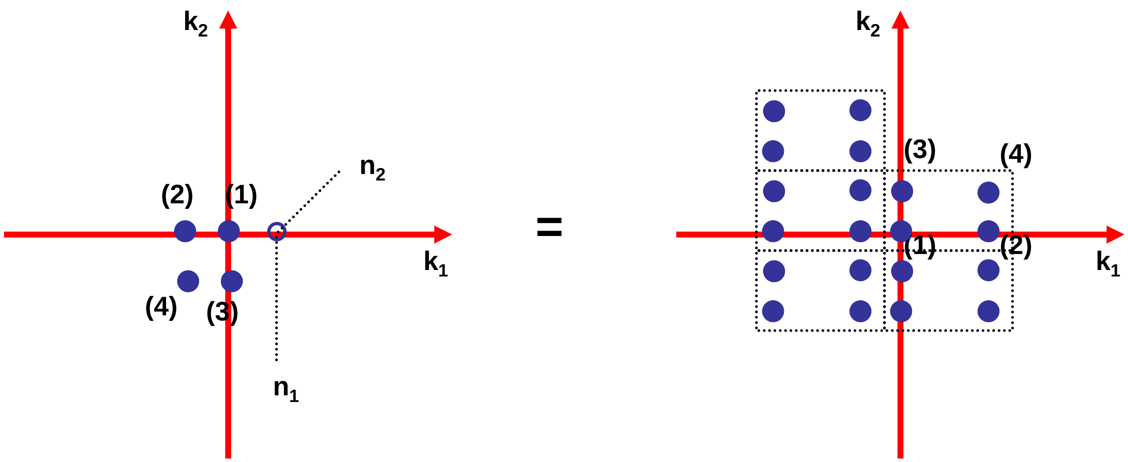

$$x[k_1,k_2] \qquad g[k_1-n_1,k_2-n_2] = h[n_1-k_1,n_2-k_2]$$

the next step would be to point-wise multiply the two signals and sum

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} x[k_1,k_2]\, h[n_1-k_1,n_2-k_2]}$$

# Periodic convolution

- this is where we depart from linear convolution
- remember that the sequences are periodic, and we really only care about what happens in the fundamental period
- we use the periodicity to fill the values missing in the flipped sequence



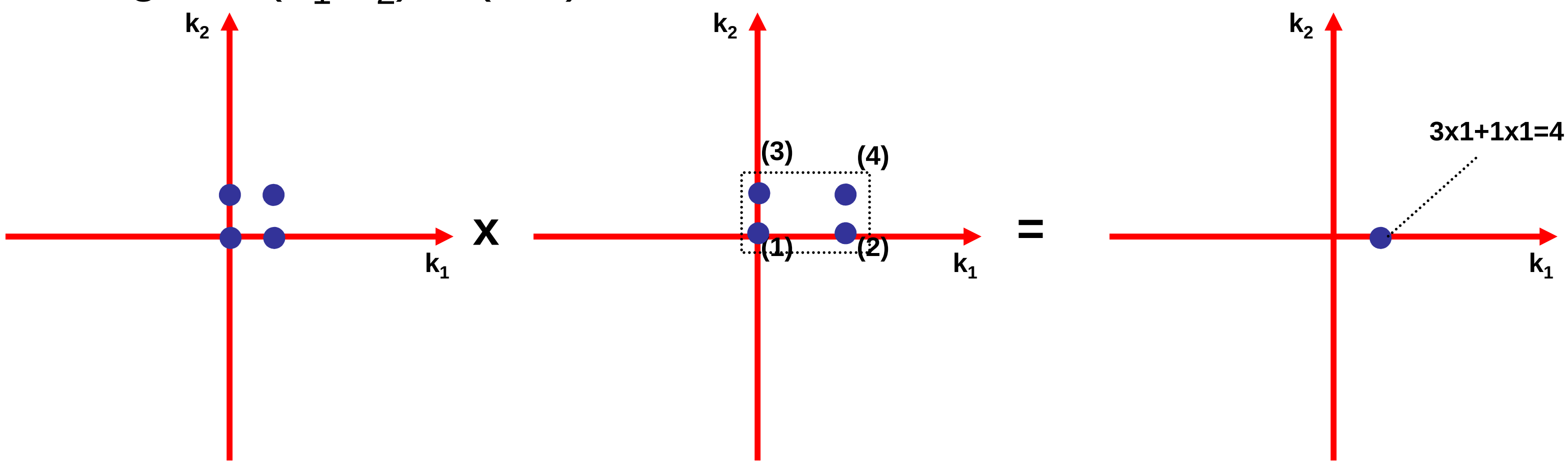

$$h[n_1 - k_1, n_2 - k_2] \qquad\qquad h[n_1 - k_1, n_2 - k_2]$$

# Periodic convolution

- **step 4):** we can finally point-wise multiply the two signals and sum

$$\boxed{x \circ h = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} x[k_1, k_2]\, h[n_1 - k_1, n_2 - k_2]}$$

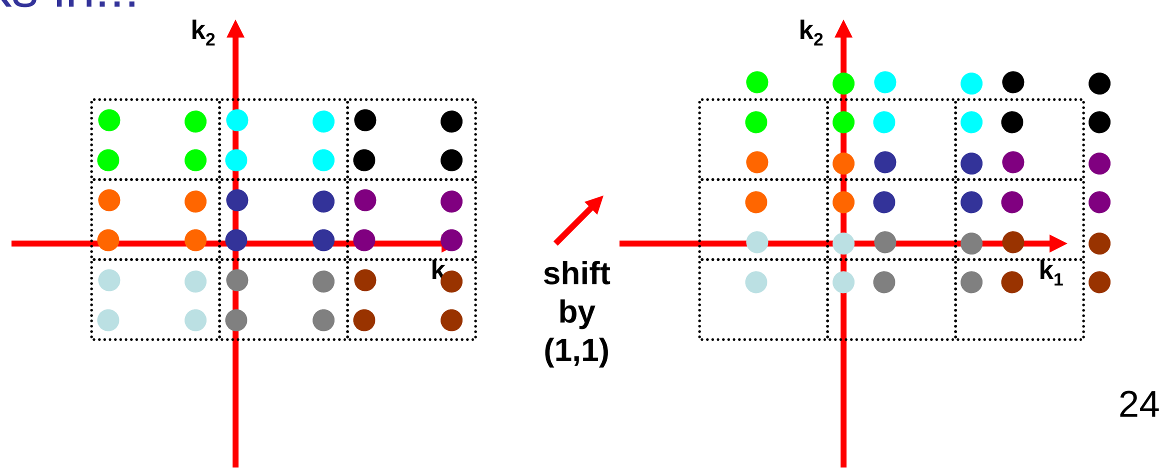
  – e.g. for $(n_1,n_2) = (1,0)$




$x[k_1, k_2]$ $\quad h[n_1 - k_1, n_2 - k_2]$ $\quad y[n_1, n_2]$

# Periodic convolution

- note: the sequence that results from the convolution is also periodic
- it is important to keep in mind what we have done
  - we work with a single period (the fundamental period) to make things manageable
  - but remember that we have periodic sequences
  - it is like if we were peeking through a window
  - if we shift, or flip the sequence we need to remember that
  - the sequence does not simply move out of the window, but the next period walks in!!!
  - note, that this can make the fundamental period change considerably



# Discrete Fourier Transform

- all of this is interesting,
  - but why do I care about periodic sequences?
  - all images are finite, I could never have such a sequence
- while this is true
  - the DFS is the easiest route to learn about the discrete Fourier transform (DFT)
- recall that the DFT is obtained by sampling the DSFT

$$\boxed{X[k_1,k_2] = X(\omega_1,\omega_2)\Big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2}}$$

  - we know that when we sample in space we have aliasing in frequency
  - well, the same happens when we sample in frequency: we get aliasing in time

# Discrete Fourier Transform

- this means that
  - even if we have a finite sequence
  - when we compute the DFT we are effectively working with a periodic sequence
- you may recall from 1D signal processing that
  - when you multiply two DFTs, you do not get convolution in time
  - but instead something called circular convolution
  - this is strange: when I flip a signal (e.g. convolution) it wraps around the sequence borders
  - well, in 2D it gets much stranger
- the only way I know how to understand this is to
  - think about the underlying periodic sequence
  - establish a connection between the DFT and the DFS of that sequence
  - use what we have seen for the DFS to help me out with the DFT

# Discrete Fourier Transform

- let's start by the relation between DFT and DFS
- the DFT is defined as

$$\boxed{X[k_1,k_2] = X(\omega_1,\omega_2)\Big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2}}$$

(here $X(\omega_1,\omega_2)$ is the DSFT) which can be written as

$$X[k_1,k_2] = \begin{cases} \displaystyle\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x[n_1,n_2]\, e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2}, & \begin{array}{l} 0 \le k_1 < N_1 \\ 0 \le k_2 < N_2 \end{array} \\ 0 & \textit{otherwise} \end{cases}$$

$$x[n_1,n_2] = \begin{cases} \displaystyle\frac{1}{N_1 N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} X[k_1,k_2]\, e^{j\frac{2\pi}{N_1}k_1 n_1} e^{j\frac{2\pi}{N_2}k_2 n_2} & \begin{array}{l} 0 \le n_1 < N_1 \\ 0 \le n_2 < N_2 \end{array} \\ 0 & \textit{otherwise} \end{cases}$$

# Discrete Fourier Transform

- comparing this

$$X[k_1,k_2]=\begin{cases}\displaystyle\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x[n_1,n_2]e^{-j\frac{2\pi}{N_1}k_1n_1}e^{-j\frac{2\pi}{N_2}k_2n_2}, & \begin{array}{l}0\le k_1<N_1\\0\le k_2<N_2\end{array}\\ 0 & otherwise\end{cases}$$

$$x[n_1,n_2]=\begin{cases}\displaystyle\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X[k_1,k_2]e^{j\frac{2\pi}{N_1}k_1n_1}e^{j\frac{2\pi}{N_2}k_2n_2} & \begin{array}{l}0\le n_1<N_1\\0\le n_2<N_2\end{array}\\ 0 & otherwise\end{cases}$$

with the DFS

$$\underline{X}[k_1,k_2]=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\underline{x}[n_1,n_2]e^{-j\frac{2\pi}{N_1}k_1n_1}e^{-j\frac{2\pi}{N_2}k_2n_2}$$

$$\underline{x}[n_1,n_2]=\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\underline{X}[k_1,k_2]e^{j\frac{2\pi}{N_1}k_1n_1}e^{j\frac{2\pi}{N_2}k_2n_2}$$

# Discrete Fourier Transform

- we see that inside the boxes

$$\boxed{\begin{aligned}0 \le k_1 < N_1 \\ 0 \le k_2 < N_2\end{aligned}} \qquad \boxed{\begin{aligned}0 \le n_1 < N_1 \\ 0 \le n_2 < N_2\end{aligned}}$$

the two transforms are exactly the same

- if we define the indicator function of the box

$$\boxed{R_{N_1 \times N_2}[n_1, n_2] = \begin{cases} 1, & \begin{aligned}0 \le n_1 < N_1 \\ 0 \le n_2 < N_2\end{aligned} \\ 0 & otherwise \end{cases}}$$

- we can write

$$\boxed{x[n_1, n_2] = \underline{x}[n_1, n_2] R_{N_1 \times N_2}[n_1, n_2]} \qquad \boxed{X[k_1, k_2] = \underline{X}[k_1, k_2] R_{N_1 \times N_2}[k_1, k_2]}$$

# Discrete Fourier Transform

- note from

$$\boxed{x[n_1,n_2] = \underline{x}[n_1,n_2]R_{N_1\times N_2}[n_1,n_2]} \qquad \boxed{X[k_1,k_2] = \underline{X}[k_1,k_2]R_{N_1\times N_2}[k_1,k_2]}$$

that working in the DFT domain is equivalent to

  - working in the DFS domain
  - extracting the fundamental period at the end

- we can summarize this as

$$\boxed{x[n_1,n_2] \underset{truncate}{\overset{periodicize}{\rightleftarrows}} \underline{x}[n_1,n_2] \overset{DFS}{\leftrightarrow} \underline{X}[k_1,k_2] \underset{periodicize}{\overset{truncate}{\rightleftarrows}} X[k_1,k_2]}$$

- in this way, I can work with the DFT without having to worry about aliasing

# Discrete Fourier Transform

$$x[n_1,n_2] \underset{truncate}{\overset{periodicize}{\rightleftarrows}} \underline{x}[n_1,n_2] \overset{DFS}{\leftrightarrow} \underline{X}[k_1,k_2] \underset{periodicize}{\overset{truncate}{\rightleftarrows}} X[k_1,k_2]$$

- this trick can be used to derive all the DFT properties
- e.g. what is the inverse transform of a phase shift?
  - let's follow the steps

$$\boxed{Y[k_1,k_2] = X[k_1,k_2]e^{-j\frac{2\pi}{N_1}k_1m_1}e^{-j\frac{2\pi}{N_2}k_2m_2}}$$

  - 1) periodicize: this causes the same phase shift in the DFS

$$\boxed{\underline{Y}[k_1,k_2] = \underline{X}[k_1,k_2]e^{-j\frac{2\pi}{N_1}k_1m_1}e^{-j\frac{2\pi}{N_2}k_2m_2}}$$

# Discrete Fourier Transform

$$x[n_1,n_2] \underset{\textit{truncate}}{\overset{\textit{periodicize}}{\rightleftarrows}} \underline{x}[n_1,n_2] \overset{DFS}{\leftrightarrow} \underline{X}[k_1,k_2] \underset{\textit{periodicize}}{\overset{\textit{truncate}}{\rightleftarrows}} X[k_1,k_2]$$

– 2) compute the inverse DFS: it follows from the properties of the DFS (page 142 on Lim) that we get a shift in space

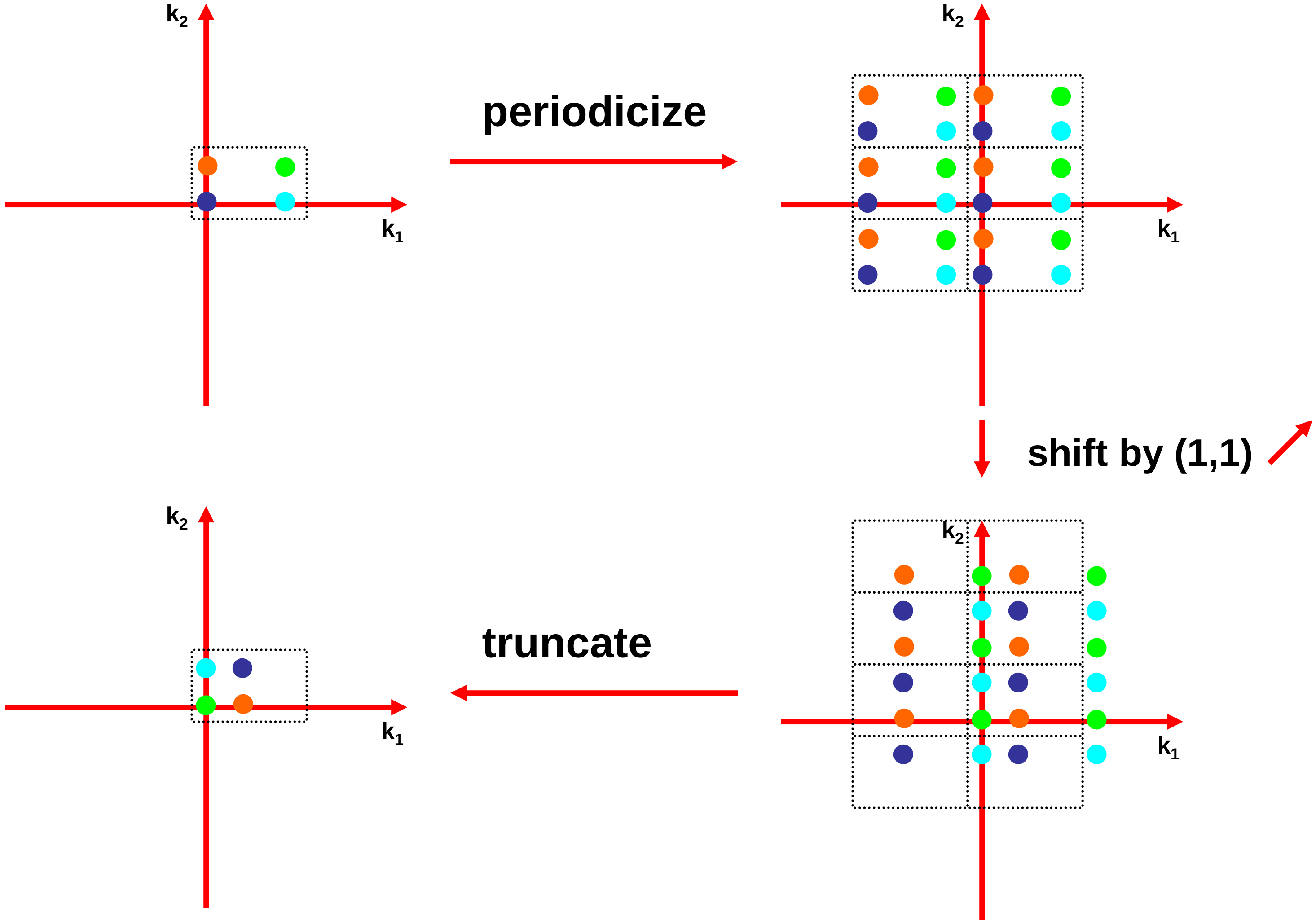
$$\boxed{\underline{y}[n_1,n_2] = \underline{x}[n_1-m_1,n_2-m_2]}$$

– 3) truncate: the inverse DFT is equal to one period of the shifted periodic extension of the sequence

$$\boxed{y[n_1,n_2] = \underline{x}[n_1-m_1,n_2-m_2]R_{N_1\times N_2}[n_1,n_2]}$$

– in summary, the new sequence is obtained by making the original periodic, shifting, and taking the fundamental period

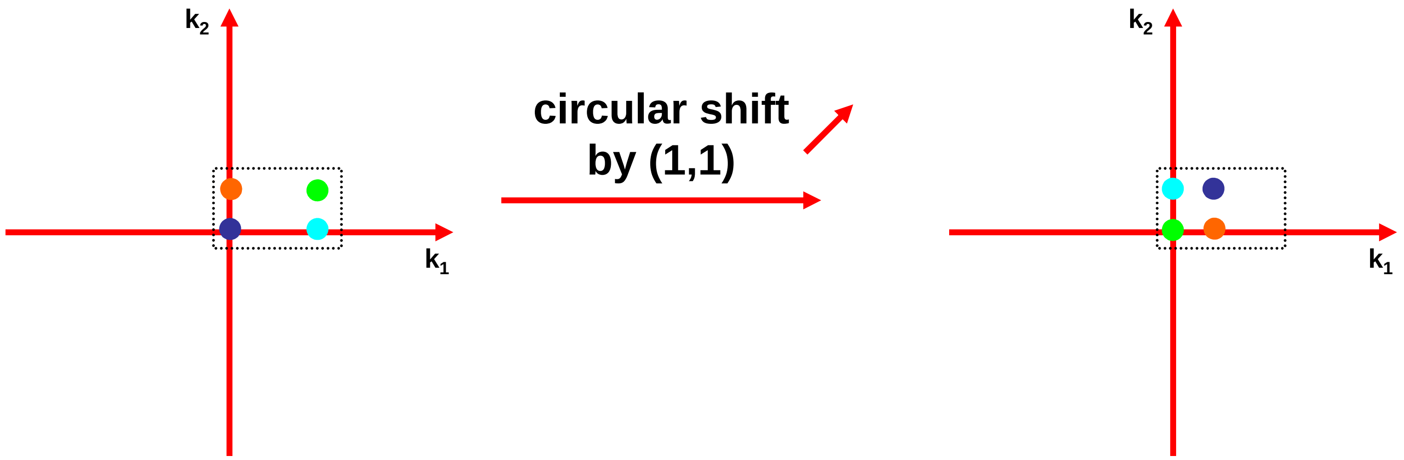
# Example




- note that what leaves on one end, enters on the other

# Example

- for this reason it is called a circular shift




- note that this is way more complicated than in 1D
- to get it right we really have to think in terms of the periodic extension of the sequence
- we will see that this shows up in the properties of the DFT,
- namely that convolution becomes circular convolution

# Any questions?