Homework Set Five

ECE 161C

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1. Suppose $\tilde{x}(n_1, n_2)$ is a periodic sequence of period $N_1 \times N_2$.

a) show that the sequence $\tilde{x}(n_1, n_2)$ is also periodic with period $2N_1 \times 2N_2$.

b) let $\tilde{X}(k_1, k_2)$ be the DFS of $\tilde{x}(n_1, n_2)$ considered as a periodic sequence of period $N_1 \times N_2$ and $\tilde{X}'(k_1, k_2)$ the DFS of $\tilde{x}(n_1, n_2)$ considered as a periodic sequence of period $2N_1 \times 2N_2$. Express $\tilde{X}'(k_1, k_2)$ in terms of $\tilde{X}(k_1, k_2)$.

c) Consider the periodic sequences of period 3×3 whose fundamental periods are defined by

$$\tilde{x}(n_1, n_2) = \begin{cases} 6, & n_1 = 0, n_2 = 2 \\ 4, & n_1 = 0, n_2 = 1 \\ 2, & n_1 = 1, n_2 = 0 \\ 3, & n_1 = 2, n_2 = 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{h}(n_1, n_2) = \begin{cases} 1, & n_1 = 0, n_2 = 0 \\ 1, & n_1 = 0, n_2 = 1 \\ 1, & n_1 = 1, n_2 = 0 \\ 1, & n_1 = 1, n_2 = 1 \\ 0, & \text{otherwise} \end{cases}$$

Compute the periodic convolution of the two sequences.

2. Consider the sequences

$$x(n_1, n_2) = \begin{cases} 6, & n_1 = 0, n_2 = 2 \\ 4, & n_1 = 0, n_2 = 1 \\ 2, & n_1 = 1, n_2 = 0 \\ 3, & n_1 = 2, n_2 = 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h(n_1, n_2) = \begin{cases} 1, & n_1 = 0, n_2 = 0 \\ 1, & n_1 = 0, n_2 = 1 \\ 1, & n_1 = 1, n_2 = 0 \\ 1, & n_1 = 1, n_2 = 1 \\ 0, & \text{otherwise} \end{cases}$$

a) consider the sequence $z(n_1, n_2)$ whose 3×3 DFT satisfies

$$Z[k_1, k_2] = X[k_1, k_2] \exp\left\{-j\left(\frac{4\pi}{3}k_1 + \frac{2\pi}{3}k_2\right)\right\}.$$

Determine $z(n_1, n_2)$.

b) determine the circular convolution $x(n_1, n_2) \odot h(n_1, n_2)$, of assumed periodicity 3×3.

c) In homework 3 we have determined the linear convolution $x(n_1, n_2) * h(n_1, n_2)$. What is the minimum size of the assumed periodicity for the circular convolution for which $x(n_1, n_2) * h(n_1, n_2) = x(n_1, n_2) \odot h(n_1, n_2)$?

d) determine $x(n_1, n_2) \odot h(n_1, n_2)$ by computing

$$IDFT[DFT[x(n_1, n_2)]DFT[h(n_1, n_2)]]$$

with the assumed periodicity of the circular convolution given by your answer in **c)**. Is your answer the same as in homework 3 **a)**? (Note: you can use MATLAB to compute DFTs and IDFTs)

3. In this problem we consider the 2D-DCT transform.

a) show that, like the DFT, the DCT can be efficiently computed by a row-column decomposition, by deriving the algorithm for this decomposition.

b) consider the sequence

$$x(n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0 \\ 2, & n_1 = 1, n_2 = 0 \\ 3, & n_1 = 0, n_2 = 1 \\ 4, & n_1 = 1, n_2 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the 2×2 DCT $X(k_1, k_2)$ directly from the definition

c) for the sequence in **b)** compute $X(k_1, k_2)$ by the row-column decomposition. Which of the two approaches is more efficient?

4. (MATLAB) The energy compaction of a transform is a measure of how much of the signal energy is compressed into the smallest possible number of transform coefficients. It is a very important property in image compression, where we want to reduce as much as possible the number of coefficients to transmit. Ideally, all the energy would be compacted into one coefficient and only that one would have to be sent. The DCT is very close to optimal in an energy compaction sense and this justifies its popularity in the compression world. In this problem we verify this energy compaction property.

We will be working with 8×8 image blocks, i.e. we view each image as a collection of 8×8 blocks. For each block we compute the discrete cosine transform (function `dct2`) and obtain an array of 8×8 frequency coefficients. The file `zz.mat` contains the rank (importance), from an energy compaction point of view, of each coefficient in the 8×8 array. The goal is to determine how many coefficients we can eliminate without affecting the quality of the transmitted image. The procedure is the following

1. read the image `lena.jpg` and the ranking matrix from the file `zz.mat` into MATLAB
2. for each non-overlapping 8×8 block in the image
   - compute the DCT coefficients using `dct2`
   - using the zz matrix, find all coefficients whose rank is greater than k and set them to zero
   - compute the inverse DCT (`idct2`) of the block and store it in the appropriate location of the reconstructed image.
3. repeat the procedure above for $k \in \{1, 4, 8, 16, 24, 32, 40, 48, 64\}$.
4. for each k compute the mean squared error

$$mse = \frac{1}{NM} \sum_{i=1}^{M} \sum_{j=1}^{N} [I(i,j) - \hat{I}(i,j)]^2$$

   between the original (I) and reconstructed ($\hat{I}$) images.

Note: The code available from the course web page implements many of these steps.

a) hand-in a plot of the reconstructed image for each k.

b) what is the smallest k which leads to a reconstructed image that is visually indistinguishable from the original? Assuming that the transmission of each coefficient requires b bits, estimate the bit savings achievable by discarding the coefficients that have no impact on the quality of the reconstructed image.

c) hand-in a plot of the mse as a function of k. Is the mse a good measure of the quality of the reconstructed image?