Analysis of Communication Systems Using Iterative Methods Based on Banach's Contraction Principle

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***Abstract*— In this paper, the application of a well known mathematical theorem, Banach's fixed point theorem [1], is investigated in iterative signal processing in communications. In most practical communication systems some sort of a contraction mapping is used to enhance the operation of the system. Thus, using a suitable iterative approach, one can set the system in its fixed point and hence, the distortion produced in the transmitter, channel and the receiver can be compensated. In other words, a loosely designed transceiver can be enhanced by an iterative method. In order to verify the truth of the proposed iterative method, the distortion of A/D and D/A converters is compensated at the receiver. Although analyses are performed on a linear system, they could be generalized to non linear ones.**

***Keywords*— iterative signal reconstruction, Banach's contraction principle, system enhancement**

I. Introduction

Communications and mathematics are closely related in many ways. Almost all the phenomena in the communication world can be modeled mathematically. In fact mathematics provides all the essential tools needed to model engineering systems. Whenever new mathematical ideas have been brought to the realm of engineering, drastic achievements have been made. Among different areas in communications, digital signal processing is more closely associated with mathematics. Different mathematical theorems may be useful for solving different engineering problems. Among all these theorems we found Banach's fixed point theorem (also known as Banach's contraction principle) to be extremely practical in signal recovery problems. A fixed point of an operator f is a solution of the equation $f(x) = x$.

One of the major aims of communications is to transmit a signal through a channel and to receive the undistorted signal at the receiver. Unfortunately distortion is inevitable in most transmitter-receiver systems. Thus, a distorted version of the signal, say $\mathbf{y} = f(\mathbf{x})$, is received instead of $\mathbf{x}$. The problem is how to estimate $\mathbf{x}$ from $\mathbf{y}$ having the model f of the system which introduced distortion to the signal. Through this paper we will show how this problem can be recast as a fixed point problem and solved using Banach's fixed point theorem. This point of view can be utilized in many practical applications including modulation-demodulation [2], signal reconstruction from non-uniform samples [3], analog to digital conversion [4], digital down conversion [5], OFDM clipping noise suppression [6], etc. In all mentioned applications, the goal is to set a system in its fixed point in order to remove distortions. Of course, the mapping D which models the system should satisfy the contraction condition so that the existence of a fixed point is guaranteed. But we seldom have to worry about this condition, because real world systems inherently have the contraction property.

From a more general point of view, a communication system can be loosely designed but an undistorted signal can be received at the receiver using signal processing techniques to set the system at its fixed point. But the major question is how to set a system at its fixed point. For instance, suppose that $\mathbf{x}$ is a signal of interest and $\mathbf{y} = f(\mathbf{x})$ is a distorted version of $\mathbf{x}$ (f is the operator which models the system). It can be easily shown that if $\mathbf{x}$ satisfies a constraint equation $\mathbf{x} = g(\mathbf{x})$, then the identity,

$$\mathbf{x} = g(\mathbf{x}) + \lambda\left[f(\mathbf{x}) - f\big(g(\mathbf{x})\big)\right]$$

holds. The solution to this equation will be the unknown signal $\mathbf{x}$. This leads to a recursive signal reconstruction algorithm such as [7],

$$\mathbf{x}_{i+1} = g(\mathbf{x}_i) + \lambda\left[f(\mathbf{x}_i) - f\big(g(\mathbf{x}_i)\big)\right]$$

In this paper we will show that under certain conditions (if f is a contraction mapping) this algorithm converges to the fixed point of g and hence, the desired signal is attained.

II. Mathematical Concepts and Definitions

In this section we will briefly review the basic mathematical concepts used in the forthcoming sections of the paper. We will start with some mathematical definitions. These definitions allow us to look at signals and systems from a more abstract point of view. Signals will be considered elements (vectors) in a function space (Banach space) and systems will be considered operators on this space. Because digital signals are vectors of finite length we can analyze digital systems in $\mathbb{R}^n$, such an analysis is not possible for analog systems. This point of view will be more developed in the next section.

Definition 1 (Banach space): A Banach space is a complete normed vector space.

Definition 2 (Contraction): A contraction of a metric space M, is a mapping $f : M \longrightarrow M$ such that for some positive constant $k < 1$ and all $x, y \in M$,

$$d(f(x), f(y)) \leq kd(x, y),$$

where d is the distance operator of the metric space. If M is a normed space, the above condition can be stated as,

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq k\|\mathbf{x} - \mathbf{y}\|$$

Furthermore, if f is a linear mapping it can be represented by a matrix T in vector space M, then T is a contraction if,

$$\|T\mathbf{x} - T\mathbf{y}\| \leq k\|\mathbf{x} - \mathbf{y}\|$$

Since T represents a linear transformation, the above condition can be reduced to,

$$\|T\mathbf{x}\| \leq k\|\mathbf{x}\|$$

If the matrix A represents a mapping on a normed vector space V, then a matrix norm can be defined by,

$$\|A\| \overset{\text{def}}{=} \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\mathbf{x}\neq 0} \left\| A\frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| = \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

If the norm on the vector space is the Euclidean norm, then the matrix norm defined above is called spectral norm and denoted $\|A\|_2$. It can be shown that the spectral norm of a matrix is equal to,

$$\|A\|_2 = \sqrt{\rho(A^\dagger A)}$$

where $A^\dagger$ is the conjugate transpose of A, and $\rho(A)$ denotes the spectral radius of A. The spectral radius of an n by n matrix T with eigenvalues $\{\lambda_i\}_{i=1}^n$ is defined by,

$$\rho(T) \overset{\text{def}}{=} \max_{0\leq i\leq 1} |\lambda_i|$$

Many other matrix norms can be defined on a normed space but we usually use the first norm defined above. Using this matrix norm, we conclude that if T is a contraction of a vector space V, its norm is bounded by a number $k < 1$,

$$\|T\| = \sup_{\mathbf{x}\neq 0} \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} \leq k$$

The above inequality yields the condition on T for being a contraction in a straightforward way. This condition on T can be easily checked but the problem is, in communications we usually have to deal with nonlinear systems. If we look at the signals as elements of a function space V, a system will be a mapping f from this space onto itself and this mapping is usually nonlinear for communication systems. Thus we seek a condition on the nonlinear mapping f that can be easily checked and yields the condition of contraction. For analog systems probably no such condition can be found and condition of contraction must be directly checked for them, but for digital systems a condition that can be more easily checked does exist. To derive this condition we need some more definitions.

Suppose $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a vector field, if $\mathbf{F}$ is differentiable at a point $\mathbf{p}$ in $\mathbb{R}^n$, its derivative is given by its Jacobian matrix $J_{\mathbf{F}}(\mathbf{p})$, and a linear approximation of $\mathbf{F}$ near $\mathbf{p}$ is given by,

$$\mathbf{F}(\mathbf{x}) \simeq \mathbf{F}(\mathbf{p}) + J_{\mathbf{F}}(\mathbf{p}).(\mathbf{x} - \mathbf{p})$$

It can be shown that if a vector field $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable on a subset S of $\mathbb{R}^n$ and $\|J_{\mathbf{F}}(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in S$, then $\mathbf{F}$ satisfies the Lipschitz condition for all $\mathbf{x}, \mathbf{y} \in S$,

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\|$$

Now we are ready to find a condition for a nonlinear digital system which guarantees that the mapping $f : V \longrightarrow V$ which represents the system on the function space V (the set of all signals) is a contraction. A digital signal can be considered a vector of finite length, therefore the function space V mentioned above can be considered to be $\mathbb{R}^n$ and the system to be a mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. For example, if we want to estimate a signal from a distorted version at the receiver, the mapping f would be a model of the transmission system. We can assume that the received distorted signals do not vary greatly from the original signal $\mathbf{x_0}$. In other words if we denote the received signal by $\mathbf{x}$, we can assume that $r = \|\mathbf{x} - \mathbf{x_0}\|$ is small. Thus the mapping f can be linearly approximated near $\mathbf{x}_0$ by,

$$f(\mathbf{x}) \simeq f(\mathbf{x}_0) + J_f(\mathbf{x}_0).(\mathbf{x} - \mathbf{x_0})$$

The derivative of f, that is $J_f(\mathbf{x})$, can be considered to be constant in the subset of $\mathbb{R}^n$ where our signals lie, so we have $J_f(\mathbf{x}) \simeq J_f(\mathbf{x}_0)$. Now suppose that we somehow know the mapping f satisfies the following inequality,

$$\left\|J_f(\mathbf{x}_0).(\mathbf{x} - \mathbf{x_0})\right\| \leq k\|\mathbf{x} - \mathbf{x_0}\|$$

for some positive constant $k < 1$ and for all signals of our interest (all distorted signals). In fact we are supposing that we know $\left\|J_f(\mathbf{x}_0)\right\| \leq k$, where $\left\|J_f(\mathbf{x}_0)\right\|$ is the matrix norm of $J_f(\mathbf{x}_0)$. This implies that,

$$\left\|J_f(\mathbf{x}).(\mathbf{x} - \mathbf{x_0})\right\| \leq k\|\mathbf{x} - \mathbf{x_0}\|$$

Thus,

$$\left\|J_f(\mathbf{x})\right\| \leq k$$

This means that the norm of the derivative of f is less than or equal to a positive constant $k < 1$, in a subset of $\mathbb{R}^n$ which is of our interest, therefore as mentioned before f satisfies the Lipschitz condition for all points in this subset,

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq k\|\mathbf{x} - \mathbf{y}\|$$

Since $k < 1$, the above inequality means that f is a contraction of the subset of $\mathbb{R}^n$ that the received signals lie in. Now we have found the condition we were looking for. As a summary of the above discussion, we can say, if f is a mapping in $\mathbb{R}^n$ that models a nonlinear digital system and $\mathbf{x}_0$ is the original undistorted signal which was fed to the system, then f is a contraction of the set of received (distorted) signals in $\mathbb{R}^n$ if,

$$\|J_f(\mathbf{x}_0)\| \leq k$$

for some positive constant $k < 1$. We will see in the next section why it is important for us to know that the function which models the system is a contraction mapping.

III. Iterative Methods Based on Banach's Contraction Principle

A. Banach's Contraction Principle

In this section we will first introduce Banach's contraction principle (also known as Banach's fixed point theorem), then we will discuss some iterative methods for signal recovery and prove their convergence using Banach's fixed point theorem.

Theorem 1 (Banach's fixed point theorem): Suppose that $f : M \longrightarrow M$ is a contraction and the metric space M is complete. Then f has a unique fixed point p and for any $x \in M$,

$$\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f \circ f \circ \cdots \circ f(x) = p$$

A simple and beautiful proof of Banach's contraction principle can be found in [8].

Banach's fixed pint theorem is a basis for many iterative methods used for signal reconstruction. Any communication system can be modeled by a mapping,

$$\mathbf{y} = G(\mathbf{x})$$

where $\mathbf{y}$ is the received distorted signal. We seek to reconstruct the original signal $\mathbf{x}$ from $\mathbf{y}$, thus we have to solve the equation $\mathbf{y} = G(\mathbf{x})$. If we define the function $\Phi(\mathbf{x})$ by,

$$\Phi(\mathbf{x}) = \mathbf{x} + G(\mathbf{x}) - \mathbf{y}$$

It is easily verified that the desired signal $\mathbf{x}$ is the fixed point of $\Phi(\mathbf{x})$. Thus if we know that $\Phi(\mathbf{x})$ is a contraction mapping, according to Banach's contraction principle the sequence below will converge to $\mathbf{x}$,

$$\mathbf{x}_{n+1} = \Phi(\mathbf{x}_n)$$

As we know from the previous section, in order for $\Phi(\mathbf{x})$ to be a contraction mapping, the below inequality must hold for some $k < 1$,

$$\|J_\Phi(\mathbf{x})\| = \|I - J_G(\mathbf{x})\| \leq k$$

This is the basis of a large class of iterative methods used for signal reconstruction.

B. An Application of Banach's Contraction Principle in the Analysis of a Linear System – Interpolation Distortion

A/D and D/A converters are widely used in transmitters and receivers. Any D/A converter uses an interpolation function. A complete discussion on interpolation functions can be found at [9]. Better interpolation functions introduce less distortion to the signal at the expense of more complexity and cost of the D/A converter. Simple and inexpensive D/A converters can be enhanced using iterative methods. The simplest interpolation function used in D/A converters is zero order hold also known as sample and hold. We are now going to discuss how the above method can be utilized to remove the distortion caused by a sample-and-hold system. Figure 1(a) shows the block diagram of a sample and hold system.

This system can be considered a function $G: \mathbb{R}^P \longrightarrow \mathbb{R}^P$, which maps the input vector $\mathbf{x}$ of length P to the output vector $\mathbf{y}$ of the same length. In order to compensate the distortion from the sample-and-hold system of Figure 1 we design the compensator system shown in Figure 1(b).

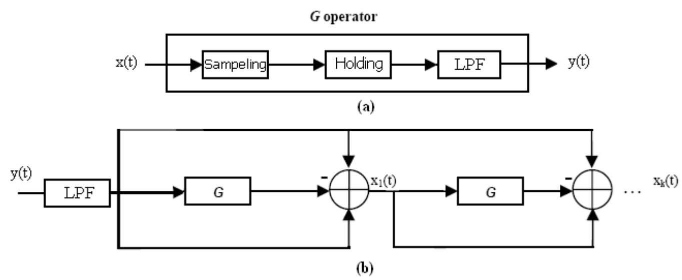

Figure 1. (a) Block diagram of a sample and hold system (b) Block diagram of a compensator system utilizing an iterative method

To analyze this system we focus our attention on how the spectrum of $\mathbf{x}$ changes while it passes through the system. Suppose that we sample the signal $x[n]$ with a rate of m. In order to avoid aliasing in the frequency domain supposing the bandwidth of $x[n]$ is L and using P point DFT the following condition must be satisfied,

$$m(2L+1) < P$$

The sampled signal is $x_s[n] = x[n]s[n]$, where $s[n]$ is an impulse train,

$$s[n] = \begin{cases} 1 & n = mk,\ \ k \in \mathbb{Z} \\ 0 & otherwise \end{cases}$$

Thus, the P point DFT of the signal $x_s[n]$ would be,

$$X_s[k] = X_s\left(e^{j\frac{2\pi}{P}k}\right) = \frac{1}{m}\sum_{i=0}^{m-1} X\left(e^{j\left(k\frac{2\pi}{mP} - i\frac{2\pi}{m}\right)}\right)$$

Therefore, the spectrum of $x_s[n]$ is an m-periodized version of the spectrum of $x[n]$, attenuated by a factor of $\frac{1}{m}$. Now if we pass the signal $x_s[n]$ through the holding block we will get,

$$x_h[n] = x_s[n] * h[n]$$

where $h[n]$ is the discrete pulse defined by,

$$h[n] = \begin{cases} 1 & -(m-1)/2 \leq n \leq (m-1)/2 \\ 0 & otherwise \end{cases}$$

and its discrete Fourier transform is given by,

$$H[k]=\frac{1}{m}\frac{\sin\left(\frac{\pi mk}{P}\right)}{\sin\left(\frac{\pi k}{P}\right)}$$

Thus, the DFT of $x_h[n]$ can be expressed as,

$$X_h[k]=X_s[k].H[k]=\frac{1}{m}\sum_{i=0}^{m-1}X\left(e^{j\left(k\frac{2\pi}{mP}-i\frac{2\pi}{m}\right)}\right).H[k]$$

The P point DFT of the lowpass filter is given by,

$$H_{LP}[k]=\begin{cases}1 & (P-L)/2\le k\le (P+L)/2\\ 0 & otherwise\end{cases}$$

Thus, when $x_h[n]$ is passed through the lowpass filter we get $y[n]$ with the P point DFT expressed by,

$$Y[k]=\begin{cases}X[k].H[k] & (P-L)/2\le k\le (P+L)/2\\ 0 & otherwise\end{cases}$$

This equality can be represented in matrix form as,

$$\underline{Y}=G(\underline{X})=W\underline{X}$$

where,

$$W=\begin{bmatrix}0 & \cdots & 0\\ \vdots & D & \vdots\\ 0 & \cdots & 0\end{bmatrix}_{P\times P}$$

where D is a diagonal $(2L+1)$ by $(2L+1)$ matrix,

$$D=\begin{bmatrix}H\left[\frac{P-L}{2}\right] & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & H\left[\frac{P+L}{2}\right]\end{bmatrix}_{(2L+1)\times(2L+1)}$$

and $\underline{Y}$ and $\underline{X}$ are vectors in the frequency domain.

Therefore, this system can be considered as a linear transform in the frequency domain. In order to reconstruct $\underline{X}$ from $\underline{Y}$ we make the sequence,

$$\underline{X}_{n+1}=\Phi(\underline{X}_n)=\underline{X}_n+G(\underline{X}_n)-\underline{Y}$$

We know that this sequence will converge to $\underline{X}$ if,

$$\|J_\Phi(\underline{Z})\|=\|I-J_G(\underline{Z})\|=\|I-W\|<1\quad\text{for all }\underline{Z}$$

Thus, to prove that the original signal $x[n]$ can be reconstructed by this method, all we have to do is to show that $\|I-W\|\le 1$. As we know from the previous sections $\|I-W\|$ is given by,

$$\|I-W\|=\sup_{\|\underline{Z}\|=1}\|(I-W)\underline{Z}\|$$

Because of the effect of the lowpass filter the vectors $\underline{Z}$ in the above equality can be considered to have the form,

$$\underline{Z}=\begin{bmatrix}0 & \cdots & 0 & z_1 & \cdots & z_{2L+1} & 0 & \cdots & 0\end{bmatrix}_{P\times 1}^T$$

Therefore, the matrix $I-W$ can be considered to be,

$$I-W=\begin{bmatrix}0 & \cdots & 0\\ \vdots & I-D & \vdots\\ 0 & \cdots & 0\end{bmatrix}_{P\times P}$$

Because the norm defined on our vector space is the Euclidean norm, from section 2 we know that $\|I-W\|$ can be expressed as,

$$\|I-W\|=\|I-W\|_2=\sqrt{\rho((I-W)^\dagger(I-W))}$$

We know that the eigenvalues of a diagonal matrix equal its diagonal elements, thus,

$$\rho((I-W)^\dagger(I-W))=\max_{i\le 2L}(1-\lambda(D)_i)^2=\max_{i\le 2L}(1-d_{ii})^2$$

where $d_{ii}=H\left[\frac{P-L}{2}+i-1\right]$. Because the sampling rate was greater than the Nyquist rate, all d_{ii} are samples of the main lobe of $H[k]$and therefore all of them are positive values. So,

$$\rho((I-W)^\dagger(I-W))=\max_{0\le i\le 2L}(1-d_{ii})^2<1$$

and,

$$\|J_\Phi(\underline{Z})\|=\|I-W\|=\sqrt{\rho((I-W)^\dagger(I-W))}<1$$

The same result can be drawn from an engineering point of view. The mapping $I-W$ acts as a strictly passive filter on the lowpass signal $x[n]$. If we denote $I-W$ by F, $\|I-W\|$ can be expressed as,

$$\|I-W\|=\|F\|=\sup_{\|\underline{Z}\|=1}\|F\underline{Z}\|$$

The norm of a signal $\underline{Z}$ is the square root of its power,

$$\|\underline{Z}\|=\sqrt{z_1^2+z_2^2+\cdots+z_P^2}=\sqrt{E_{\underline{Z}}}$$

Therefore, the constraint $\|\underline{Z}\|=1$ in the above equality means that the input power of the filter F is equal to 1, since the filter is passive its output power $E_y$will be less than unity, thus,

$$\|J_\Phi(\underline{Z})\|=\|I-W\|=\|F\|=\sup\|\underline{Z}\|=\sup\sqrt{E_{\underline{Z}}}<1$$

Therefore, $\Phi(\underline{X})$ is a contraction mapping and the distortion of the sample and hold system can be compensated by the abovementioned method. It should be noted that the rate of convergence of the iterative method is inversely dependent on $\|J_\Phi(\underline{Z})\|$,

$$\text{Rate of Convergence}\propto\frac{1}{\|J_\Phi(\underline{Z})\|}=\frac{1}{max_{0\le i\le 2L+1}(1-d_{ii})^2}$$

The same method can be used to compensate the distortion introduced to the signal from a first order hold system. The same argument applies to this case except that the expression for $H[k]$ should be replaced by,

$$H[k]=\left(\frac{1}{m}\frac{\sin\left(\frac{\pi mk}{P}\right)}{\sin\left(\frac{\pi k}{P}\right)}\right)^2$$

Since $d_{ii} < 1$ so $(1 - d_{ii}^2)^2 > (1 - d_{ii})^2$, if we use the same method to remove the distortion of a first order hold system, the rate of convergence would be less that that of a zero order.

C. An Application of Banach's Contraction Principle in the Analysis of a Nonlinear System – Feedback Compensator

As a second application of Banach's contraction principle in system analysis, we are going to analyze feedback systems using this theorem. In most of the control systems we like the output waveform to be as similar as possible to the input waveform. Engineers use feedback as a very powerful tool to make the output waveform of a system follow the input waveform. Feedback is used in different structures, but here we analyze the simplest one and prove the convergence of the output of a feedback system using Banach's contraction principle. Imagine a physical system which is modeled by a function $G: \mathbb{R} \longrightarrow \mathbb{R}$ (Figure 2(a)).

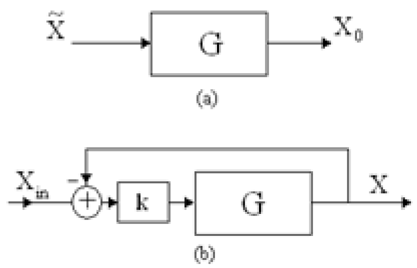

Figure 2. (a) Block diagram of a model of a physical system (b) Block diagram of the closed loop feedback system of the system in part (a)

The output of the system due to an input $\tilde{x}$ is x_0. Using the system modeled by G, we make the closed loop system of figure 2(b). For the feedback system we have,

$$x = \phi(x) = k.G(x_{in} - x)$$

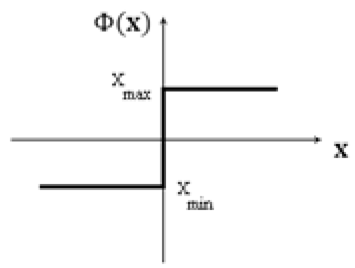

Figure 3. $\phi(x)$versus the input signal

It can be seen that for the case of $k \longrightarrow \infty$ the fixed point of the mapping $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ is $x = x_{in}$. Therefore, according to Banach's contraction principle the condition $\|J_\phi(x)\| \leq 1$ guarantees that the system will converge to its fixed point and the output waveform will follow the input waveform. Considering the delays in the feedback loop, we can apply the concept of iteration to the system, and the output of the system can be thought to be a series (although it does not change discretely) which converges to $\mathbf{x}_{in}$. From a mathematical point of view we have,

$$x_i = k.G(x_{in} - x_{i-1})$$

where,

$$x_i = x(i.\Delta t) \ \ , \ \ \Delta t \longrightarrow 0$$

It should be noted that in general G is a nonlinear operator. Thus, this is an application of Banach's contraction principle in nonlinear systems. Because of the saturation phenomenon in systems the mapping ϕ can be considered to be the function of figure 3. For ϕ to be a contraction mapping we must have,

$$\|\phi(x) - \phi(y)\| < \|x - y\|$$

Because the values of x and y are always both negative or both positive (convergence is one-sided), the right side of the above equation would be always zero. Thus the above equation holds and ϕ is a contraction mapping. As it was mentioned before, the output of the system changes continually and because of the stability of the system, the output will not oscillate between x_{max} and x_{min}. This is another reason for the right side of the above equation to be always zero. A feedback cannot be directly applied to wireless communication systems, therefore we usually use the concept of feedback in another way. The behavior of the system from transmitter to receiver is modeled by a transfer function G. Then, the feedback system ϕ is modeled in the receiver by making a feedforward series using G. In practice, a few number of iterations would be enough to enhance the behavior of the system. This method has been proved to be quite useful in nonuniform and missing sampling compensation [10], OFDM clipping and CDMA interference cancellation systems [11].

IV. SIMULATION RESULTS

We simulated the behavior of the compensator system of Figure 1. As it was mentioned in the previous section, the rate of convergence of the method would be greater when applied to a zero order hold system compared to that of a first order hold. This fact is verified in Figures 4 and 5. From these figures it can be seen that increasing the OSR (Over Sampling Ratio) causes the output of the system to converge more rapidly. This fact can be verified by the relation between the rate of convergence and $\|J_\Phi(\underline{Z})\|$ mentioned in the previous section. As the OSR increases the value of low frequency components of $H[k]$ increase; thus, $H[k]$ becomes more flat in the bandwidth of the lowpass filter, this causes $\|I - W\|$ to decrease,

$$\|I - W\| = \max_{0 \leq i \leq 2L} (1 - d_{ii})^2 = \max_{(P-L)/2 \leq k \leq (P+L)/2} (1 - H[k])^2$$

and so will $\|J_\Phi(\underline{Z})\|$, resulting in a greater rate of convergence.

Another point that can be seen on Figures 4 and 5 is that the rates of convergence of the method for the two systems become nearer to each other as the OSR increases.

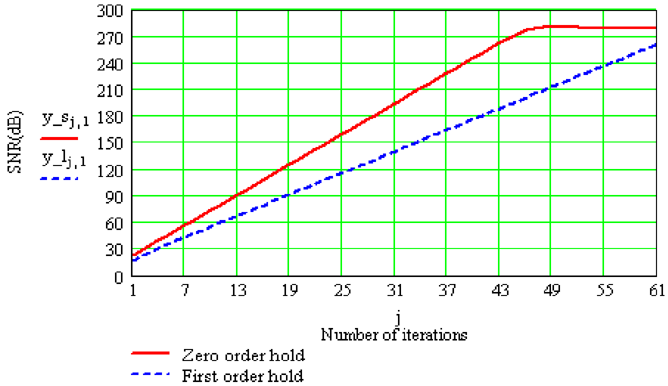

Figure 4. SNR_{out}vs. number of iterations for a sample and hold system and a first order hold system for OSR = 1.25

Figure 5. SNR_{out}vs. number of iterations for a sample and hold system and a first order hold system for OSR = 2.5

According to Figure 6 as the OSR increases the values of the low frequency components of the functions $\mathrm{sinc}\, k$ and $\mathrm{sinc}^2\, k$ become almost the same. Therefore $\|I - W\|$would be the same for the two systems and the method would converge with the same rate.

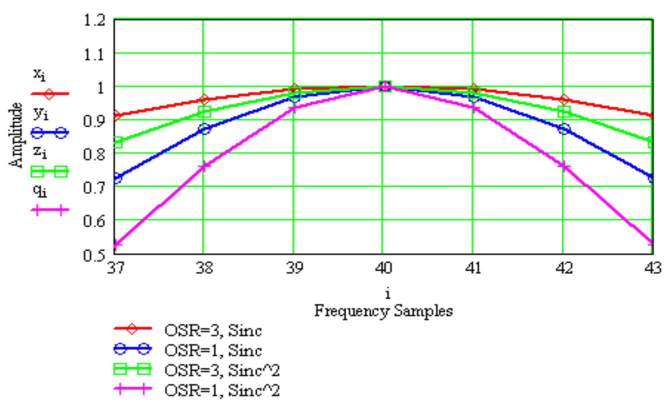

Figure 6. $\mathrm{sinc}(k)$ and $\mathrm{sinc}^2(k)$ for different OSRs

The output of a feedback system is shown in Figure 7. The result clearly shows the convergence of the output of the system and thus, verifies the truth of the argument on feedback systems in the previous section.

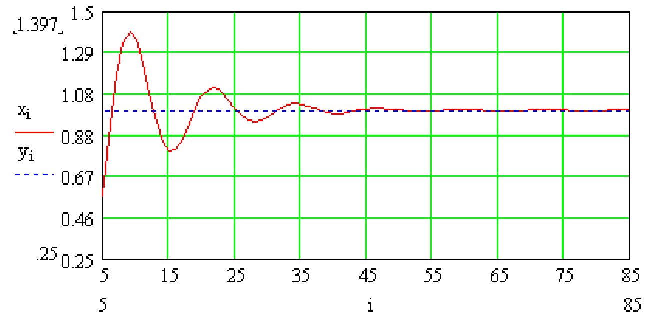

Figure 7. The output of a feedback system versus time

V. CONCLUSION

In summary, a vast class of signal processing problems such as audio and Image reconstruction, linear and nonlinear distortion compensation in transceivers, etc. can be considered fixed point problems can be consideredand solved using fixed point theorems such as Banach's contraction principle. Fortunately, most of communication systems satisfy the contraction constraint. Therefore, iterative approaches can be utilized to solve these problems. In this paper, we analyzed some simple applications of this method and simulated them. However, there are several more applications beyond those presented here [12]. Now, we are attempting to enhance receivers' performance in the 4G communication systems using this method.

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